Midterm Exam

P571

October 1, 2009

SHOW ALL WORK TO GET FULL CREDIT!

PART I:**ONLY TWO OF THE FOUR PROBLEMS WILL BE GRADED**. Take a look at the 4 problems. Each of them is worth 25 points. To make sure that you have enough time to do your work you will have to turn in only 2 of the 4 problems. **If you turn more than 2 problems only the two on top will be graded and 5 points will be deducted from your grade.**
PART II: Take the test home and bring **ALL** the problems solved on Tuesday October 6. Your grade for the test will be the **sum of the two** parts. A perfect score is worth 150 points.

Problem 1: Knowing that

$$\mathrm{F}^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a) Calculate $\mathrm{F}_a{}^b$ and $\mathrm{F}^a{}_b$.(10 points)
b) Calculate explicitely $\mathrm{F}_a{}^b\mathrm{F}^a{}_b$. Is it a tensor? If so what is its rank? (10 points)
c) What is the rank of $\mathrm{F}_a{}^b\mathrm{F}^c{}_b$? Provide the value of $\mathrm{F}_1{}^b\mathrm{F}^2{}_b$.(5 points)

Problem 2:
a) Write and prove in tensor notation the Lagrange identity given by

$$(\mathbf{A} \times \mathbf{B}).(\mathbf{C} \times \mathbf{D}) = (\mathbf{A}.\mathbf{C})(\mathbf{B}.\mathbf{D}) - (\mathbf{A}.\mathbf{D})(\mathbf{B}.\mathbf{C}).$$

(15 points)
b) Using (a) provide an expression for $(\mathbf{A} \times \mathbf{B}).(\mathbf{A} \times \mathbf{B})$. Under what conditions is it 0? (10 points)

Problem 3:
a) A charge Q is located at $\mathbf{r}' = (2, -1, 1)$. Using delta functions in cartesian coordinates provide an expression for the density of charge $\rho(\mathbf{r})$, and verify that $\int_{\text{all space}} \rho(\mathbf{r})d\mathbf{r} = Q$. (9 points)
b) A charge Q is located at $\mathbf{r}' = (2, -1, 1)$. Using delta functions in cylindric coordinates provide an expression for the density of charge $\rho(\mathbf{r})$, and verify that $\int_{\text{all space}} \rho(\mathbf{r})d\mathbf{r} = Q$. (8 points)
c) A charge Q is uniformly distributed on a circular ring of radius R. Using delta functions provide an expression for the density of charge $\rho(\mathbf{r})$, and verify that $\int_{\text{all space}} \rho(\mathbf{r})d\mathbf{r} = Q$. (8 points)

Problem 4:

a) The magnetic induction $\mathbf{B}$ is defined by the Lorentz force equation,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}).$$

Carrying out three experiments, we find that if

$$\mathbf{v} = \hat{\mathbf{x}}, \qquad \frac{\mathbf{F}}{q} = 2\hat{\mathbf{z}} - 4\hat{\mathbf{y}},$$

$$\mathbf{v} = \hat{\mathbf{y}}, \qquad \frac{\mathbf{F}}{q} = 4\hat{\mathbf{x}} - \hat{\mathbf{z}},$$

$$\mathbf{v} = \hat{\mathbf{z}}, \qquad \frac{\mathbf{F}}{q} = \hat{\mathbf{y}} - 2\hat{\mathbf{x}}.$$

From the results of these three separate experiments calculate the magnetic induction $\mathbf{B}$.(13 points)

b) If $\nabla \times \mathbf{u} = 0$ and $\nabla \times \mathbf{v} = 0$ find $\nabla.(\mathbf{u} \times \mathbf{v})$. You can choose to use vector analysis or work with tensor notation.(12 points)