Midterm Exam

P571

October 5, 2010

SOLUTION:

Problem 1:

a)

$$E^{'i} = (\cos\phi E_x + \sin\phi E_y, -\sin\phi E_x + \cos\phi E_y, E_z).$$

b)

$$F^{'\gamma\delta} = M^{\gamma}{}_{\alpha} M^{\delta}{}_{\beta} F^{\alpha\beta}.$$

c) We know that $F^{\alpha\beta} \neq 0$ if $\alpha = 0$ or $\beta = 0$. Then,

$$F^{'\gamma\delta} = M^{\gamma}{}_{0} M^{\delta}{}_{\beta} F^{0\beta} + M^{\gamma}{}_{\alpha} M^{\delta}{}_{0} F^{\alpha 0},$$

but in Eq.(4) we see that $M^{\alpha}{}_{0} = \delta_{\alpha,0}$ then we obtain:

$$F^{'\gamma\delta} = \delta_{\gamma 0} M^{\delta}{}_{\beta} F^{0\beta} + M^{\gamma}{}_{\alpha} \delta_{\delta 0} F^{\alpha 0},$$

which shows that the elements of $F^{'\gamma\delta} \neq 0$ are $F^{'0\delta}$ and $F^{'\gamma 0}$ with γ, δ= 1, 2, or 3.

d) Now let's calculate explicitely the non-zero elements of $F^{'\gamma\delta}$:

$$F^{'01} = M^{1}{}_{1} F^{01} + M^{1}{}_{2} F^{02} = -\cos\phi E_x - \sin\phi E_y$$

$$F^{'02} = M^{2}{}_{1} F^{01} + M^{2}{}_{2} F^{02} = \sin\phi E_x - \cos\phi E_y$$

$$F^{'03} = M^{3}{}_{3} F^{03} = -E_z$$

$$F^{'10} = M^{1}{}_{1} F^{10} + M^{1}{}_{2} F^{20} = \cos\phi E_x + \sin\phi E_y$$

$$F^{'20} = M^{2}{}_{1} F^{10} + M^{2}{}_{2} F^{20} = -\sin\phi E_x + \cos\phi E_y$$

$$F^{'30} = M^{3}{}_{3} F^{30} = E_z.$$

Comparing with the components of $E^{'i}$ obtained in (a) we get:

$$F'^{\gamma\delta} = \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & 0 & 0 \\ E'_y & 0 & 0 & 0 \\ E'_z & 0 & 0 & 0 \end{pmatrix}.$$

e) Comparing the results of (d) with (a) we see that we obtain equivalent information using both approaches, i.e., that in system S' there is a purely electric field with components $E'_x = \cos\phi E_x + \sin\phi E_y$, $E'_y = -\sin\phi E_x + \cos\phi E_y$, and $E'_z = E_z$.

Problem 2:

a) In tensor notation

$$\nabla.(\mathbf{A}\times\mathbf{B}) = \partial^i \epsilon_{ijk} A^j B^k =$$

$$\epsilon_{ijk} B^k \partial^i A^j + \epsilon_{ijk} A^j \partial^i B^k =$$

$$B^k \epsilon_{kij} \partial^i A^j + A^j \epsilon_{jki} \partial^i B^k =$$

$$B^k \epsilon_{kij} \partial^i A^j - A^j \epsilon_{jik} \partial^i B^k =$$

$$\mathbf{B}.(\nabla\times\mathbf{A}) - \mathbf{A}.(\nabla\times\mathbf{B}).$$

b) In (a) we showed that

$$\nabla.(\mathbf{A}\times\mathbf{B}) = \partial^i \epsilon_{ijk} A^j B^k.$$

Notice that the expression arises from the contraction of the indices of a tensor of rank 6 that arises from the direct product of the vectors ∂^i, A^j and B^k and the pseudotensor ϵ_{ijk}. Since all the indices are contracted the result is a tensor of rank 0, i.e., an scalar.

c) Since ∂^i, A^j and B^k are vectors and ϵ_{ijk} is a pseudotensor, the tensor of rank 0 that resulted in part (b) is a pseudoscalar because it transforms as such under an inversion due to the $|a| = -1$ factor in the transformation upon inversion of ϵ_{ijk}. More explicitely:

$$\partial'^i \epsilon'_{ijk} A'^j B'^k = a^i{}_m \partial^m |a| a_i{}^r a_j{}^s a_k{}^t \epsilon_{rst} a^j{}_u A^u a^k{}_v B^v =$$

$$|a| \delta_m{}^r \delta^t{}_v \delta^s{}_u \partial^m \epsilon_{rst} A^u B^v =$$

$$|a| \partial^r \epsilon_{rst} A^r B^t.$$

d) $\nabla.(\mathbf{A}\times\mathbf{B}) = 0$ if:

$$\mathbf{A}\times\mathbf{B} = \nabla\times\mathbf{C},$$

$$\mathbf{A}\times\mathbf{B} = 0,$$

which means that **A** and **B** are parallel or antiparallel.

If

$$\mathbf{A}\times\mathbf{B} = \nabla\times\mathbf{C},$$

If **B** is perpendicular to $\nabla\times\mathbf{A}$ and **A** is perpendicular to $\nabla\times\mathbf{B}$.

If

$$\nabla\times\mathbf{A} = \nabla\times\mathbf{B} = 0.$$

If

$$\mathbf{B}.(\nabla\times\mathbf{A}) = \mathbf{A}.(\nabla\times\mathbf{B}).$$

If **A** and **B** are independent of the coordinates or if any of the them is 0 then $\nabla.(\mathbf{A}\times\mathbf{B})$ would trivially be zero.

Problem 3:

We know that

$$\delta(g(x)) = \frac{\sum_i \delta(x - x_i)}{|g'(x_i)|},$$

where $g(x_i) = 0$. Thus, in this case $g(x) = 3x^2 + x - 2$ which vanishes at $x_1 = 2/3$ and $x_2 = -1$. We see that $g'(x) = 6x + 1$, thus $g'(x_1) = 5$ and $g'(x_2) = -5$. Then

$$\delta(3x^2 + x - 2) = \frac{\delta(x - \frac{2}{3})}{|5|} + \frac{\delta(x + 1)}{|-5|}.$$

Then

$$\int_{-\infty}^{\infty} (x^2 + 18)\delta(3x^2 + x - 2)dx = \int_{-\infty}^{\infty} \frac{(x^2 + 18)}{5}[\delta(x - 2/3) + \delta(x + 1)]dx =$$

$$= \frac{1}{5}(\frac{4}{9} + 18 + 1 + 18) = \frac{337}{45}.$$

Problem 4:

a) In tensor notation we have:

$$\epsilon_{ijk} A^j \epsilon^{klm} B_l C_m = V_i,$$

thus, the expresion is a tensor of rank 1.

In order to distinguish between a tensor and a pseudotensor we know that under an inversion a tensor transforms as

$$T'_{i_1,...,i_k} = a_{i_1}{}^{j_1} ... a_{i_k}{}^{j_k} T_{j_1,...,j_k},$$

and a pseudotensor transforms as

$$S'_{i_1,...,i_k} = |a| a_{i_1}{}^{j_1} ... a_{i_k}{}^{j_k} S_{j_1,...,j_k},$$

with $|a| = -1$ for the inversion. Notice that ϵ_{ijk} is a pseudotensor, this means that if we transform V_i to V'_i via an inversion, each of the two Levi-Civita tensors contribute with a factor $|a| = -1$ so that the product of both factors is always 1. We will use this in the remaining parts of this problem.

b) If **A**, **B**, and **C** are vectors we see from the expression obtained in part (a) that under an inversion no additional factors $|a| = -1$ will appear and thus V_i transforms as a tensor.

c) If **A**, **B**, and **C** are pseudovectors we see from the expression obtained in part (a) that under an inversion three factors $|a| = -1$ will appear and thus V_i transforms as a pseudotensor.

d) If **A** and **B** are pseudovectors and **C** is a vector, we see from the expression obtained in part (a) that under an inversion two additional factors $|a| = -1$ will appear and thus V_i transforms as a tensor.