Midterm Exam #2

P571

November 9, 2010

SHOW ALL WORK TO GET FULL CREDIT!

PART I:**ONLY TWO OF THE FOUR PROBLEMS WILL BE GRADED**. Take a look at the 4 problems. Each of them is worth 25 points. To make sure that you have enough time to do your work you will have to turn in only 2 of the 4 problems. **If you turn more than 2 problems only the two on top will be graded and 5 points will be deducted from your grade.**

PART II: Take the test home and bring **ALL** the problems solved on Thursday November 11. Your grade for the test will be the **sum of the two** parts. A perfect score is worth 150 points.

Problem 1: Consider the differential equation

$$x^2y'' - 6y = 0.$$

Find $y = y(x)$ using Frobenius method along the following steps:

a) Propose an appropriate power series solution. (5 points)

b) Find the indicial equation and provide the solutions. (5 points)

c) Find the two independent solutions. (5 points)

d) Propose a general solution with arbitrary coeficients.(5 points)

e) If you had to solve the given differential equation in the interval $0 \leq x \leq 100$, what will be the form of your solution? Why? (5 points)

Problem 2: A spherical shell of radius a has a surface charge given by $\sigma(\theta) = \sigma_0 \cos^2\theta$.

a) In order to find the electrical potential produced by the charged shell in all space, what differential equation do you need to solve? Why? (5 points)

b) In terms of what functions do you expect to obtain the solution? Why? (5 points)

c) Write an expression for the solution with as few undetermined coefficients as possible. (5 points)

d) Write the boundary conditions that you will use to obtain the undetermined coefficients.(5 points)

e) Find the coefficients and provide an explicit expression for the electrical potential $\Phi(\mathbf{r})$ produced everywhere by the spherical shell. (5 points)

Problem 3: The Legendre polynomials $P_l(x)$ and the associated Legendre functions $P_l^m(x)$ are defined in the interval $-1 \leq x \leq 1$.

a) Is it possible to expand $P_l^m(x)$ in terms of $P_l(x)$ in the interval $-1 \leq x \leq 1$? Justify your answer.(5 points)

Now consider the function $f(x)$ defined in the interval $-1 \leq x \leq 1$ given by $f(x) = \sqrt{1-x^2}$.

b) Draw the function $f(x)$ and write a formal expression for its expansion in terms of Legendre polynomials. Why is it possible to expand $f(x)$ in terms of $P_l(x)$? (5 points)

c) Find explicitly the coefficients of the first 3 terms in the expansion. (5 points)

d) Now write $f(x)$ in terms of the associate Legendre functions.(5 points)

e) Explain how the results that you obtained in parts (c) and (d) relate to the question asked in part (a).(5 points)

Problem 4: Consider a charge q at $(a, 0, \sqrt{3}a)$ and a charge $-q$ at $(-a, 0, \sqrt{3}a)$ where the positions are given in Cartesian coordinates (x, y, z).

a) Draw the array of charges and write an expression for the electrical potential generated by both of them at a generic point $\mathbf{r}$ in space. (5 points)

b) Write the total potential $\phi(r, \theta, \phi)$ in terms of spherical harmonics. If you obtain two expressions indicate the values of r for which each expression is valid. (5 points)

c) Write the expression for the potential at $r = 4a$ and at $r = a$ using the results that you obtained in part b).(5 points)

Now consider the case in which a metallic shell of radius $R = a$ centered at $(0, 0, 0)$ and at potential $V = 0$ is added with the charges q and $-q$ remaining at their original locations.

d) Write an expression for the potential outside the sphere. Hint: the potential is due to the external charges and the charge arrangement on the surface of the grounded sphere. Your expression should contain only one set of constants to be determined by the boundary conditions.(5 points)

e) What is the boundary condition that would allow you to determine the value of the constants? (5 point)

BONUS POINTS!!!:Find the constants (10 bonus points)