Midterm Exam #2

P571

November 9, 2010

SOLUTION:

Problem 1:

a) We propose a power series solution of the form:

$$y(x) = \sum_{j=0}^{\infty} a_j x^{k+j}. \tag{1}$$

b) Let's calculate:

$$y'(x) = \sum_{j=0}^{\infty} a_j (k+j) x^{k+j-1}, \tag{2}$$

and

$$y''(x) = \sum_{j=0}^{\infty} a_j (k+j)(k+j-1) x^{k+j-2}. \tag{3}$$

Replacing in the differential equation we obtain:

$$\sum_{j=0}^{\infty} a_j (k+j)(k+j-1) x^{k+j} - 6 \sum_{j=0}^{\infty} a_j x^{k+j} = 0. \tag{4}$$

When $j = 0$ the lowest power of x is x^k and its coefficient vanishes if $k(k-1) = 6$. Thus, the indicial equation is then

$$k^2 - k - 6 = 0, \tag{5}$$

which is solved by $k = 3$ and $k = -2$.

c) Let's find the solutions:

For $k = 3$

$$y(x) = \sum_{j=0}^{\infty} a_j x^{3+j}. \tag{6}$$

From Eq.(4), setting $k = 3$, we find that:

$$a_j[(3+j)(2+j) - 6] = 0 \tag{7}$$

which is satisfied for an arbitrary a_0 if $j = 0$ and for $a_j = 0$ if $j \neq 0$. Thus

$$y_1(x) = a_0 x^3. \tag{8}$$

For $k = -2$

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j-2}. \tag{9}$$

From Eq.(4), setting $k = -2$, we find that:

$$a_j[(j-2)(j-3) - 6] = 0 \tag{10}$$

which is satisfied for an arbitrary a_0 if $j = 0$. Thus

$$y_2(x) = a'_0 x^{-2}. \tag{11}$$

Notice that for $j = 5$ we can set $a_5 \neq 0$ but this leads to the solution already found (Eq.(8)). For all other j's $a_j = 0$ if $j \neq 0$.

d) The general solution is given by:

$$y(x) = Ax^3 + \frac{B}{x^2}. \tag{12}$$

e) In the interval $0 \leq x \leq 100$ we need to set $B = 0$ to avoid a divergence at $x = 0$. Then the solution has the form:

$$y(x) = Ax^3. \tag{13}$$

Problem 2:

a) We need to solve Laplace's equation in two different regions defined by the charged spherical surface. We cannot work in a single region because there is charge at $r = a$ and Laplace's equation is not valid there.

b) I expect to obtain the solution in terms of powers of r and Legendre polynomials because the boundary conditions are defined on a sphere and there is azimuthal symmetry.

c) In region I ($r \leq a$) I propose:

$$\Phi^I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{14}$$

we set to zero the coefficient of negative powers of r since the potential cannot diverge at $r = 0$. In region II ($r \geq a$) I propose:

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), \tag{15}$$

where the coefficients of positive powers of r have been set to zero because the potential has to vanish as $r \to \infty$.

d) In order to determine the two sets of undetermined coefficients A_l and B_l I need two boundary conditions. We know that at $r = a$ the potential has to be continuous then:

$$\Phi^I|_{r=a} = \Phi^{II}|_{r=a}. \tag{16}$$

We also know that the normal component of the electric field across a charged surface has a jump equal to σ/ϵ_0 where σ is the surface density of charge. In this case the normal to the surface is the radial component, then $E_n = E_r = -\frac{\partial\Phi}{\partial r}$ and the second boundary condition becomes:

$$\frac{\partial\Phi^{II}}{\partial r}\Big|_{r=a} - \frac{\partial\Phi^I}{\partial r}\Big|_{r=a} = -\frac{\sigma_0 \cos^2\theta}{\epsilon_0}. \tag{17}$$

e) From Eq.(16) we find that

$$A_l = \frac{B_l}{a^{2l+1}}. \tag{18}$$

And from Eq.(17) we obtain:

$$\sum_{l=0}^{\infty}[-(l+1)\frac{B_l}{a^{l+2}} - lA_l a^{l-1}]P_l(\cos\theta) = -\frac{\sigma_0 \cos^2\theta}{\epsilon_0}. \tag{19}$$

Multiplying both sides of Eq.(19) by $P_m(\cos\theta)$ and integrating over $\cos\theta$ in the interval $[-1,1]$ we obtain:

$$\frac{2}{(2l+1)}\delta_m, l[-(l+1)\frac{B_l}{a^{l+2}} - lA_l a^{l-1}] = -\frac{\sigma_0}{\epsilon_0}(\frac{2}{3}\delta_{m,0} + \frac{4}{15}\delta_{m,2}). \tag{20}$$

The right hand side arizes from the expansion of $\cos^2\theta$ in terms of the Legendre polynomials. Only polynomials with l even are present because the function to be expanded is even, and only polynomials with $l \leq 2$ are going to be present because higher order polynomials contain powers of $\cos\theta$ higher than 2. Replacing Eq.(18) in Eq.(20) we get:

$$\frac{2}{(2m+1)}(2m+1)\frac{B_m}{a^{m+2}} = \frac{\sigma_0}{\epsilon_0}(\frac{2}{3}\delta_{m,0} + \frac{4}{15}\delta_{m,2}). \tag{21}$$

Then, if $m \neq 0$, or $m \neq 2$ then $A_m = B_m = 0$. If $m = 0$

$$B_0 = \frac{\sigma_0 a^2}{3\epsilon_0}, \tag{22}$$

and

$$A_0 = \frac{\sigma_0 a}{3\epsilon_0}. \tag{23}$$

If $m = 2$

$$B_2 = \frac{2\sigma_0 a^4}{15\epsilon_0}, \tag{24}$$

and

$$A_2 = \frac{2\sigma_0}{15a\epsilon_0}. \tag{25}$$

Replacing in Eq.(14) and (15) we obtain:

$$\Phi^I(r,\theta) = \frac{\sigma_0}{3\epsilon_0}(a + \frac{2}{5}\frac{r^2}{a}P_2(\cos\theta)), \tag{26}$$

and

$$\Phi^{II}(r,\theta) = \frac{a^2\sigma_0}{3r\epsilon_0}(1 + \frac{2}{5}\frac{a^2}{r^2}P_2(\cos\theta)). \tag{27}$$

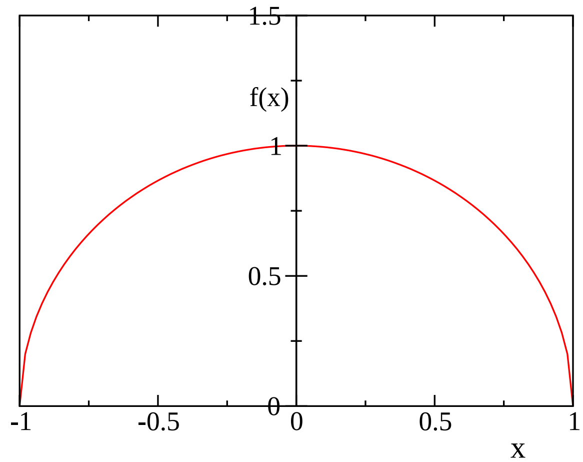
Problem 3:

a) We can expand $P_l^m(x)$ in terms of Legendre polynomials because they form a set of orthogonal functions in the interval $[-1,1]$ in which the well-behaved $P_l^m(x)$ are defined.

b)

A formal expression for the expansion is given by

$$f(x) = \sqrt{1-x^2} = \sum^{\infty} a_l P_l(x). \tag{28}$$



The function can be expanded in terms of $P_l(x)$ because it is well behaved function in the $[-1, 1]$ interval.

c) Using orthogonality of the Legendre polynomials we find that

$$\int_{-1}^{1} P_m(x)\sqrt{1-x^2}dx = \frac{2}{2l+1}a_l\delta_{l,m}. \tag{29}$$

For $l = 0$ I obtain $a_0 = \frac{\pi}{4}$.

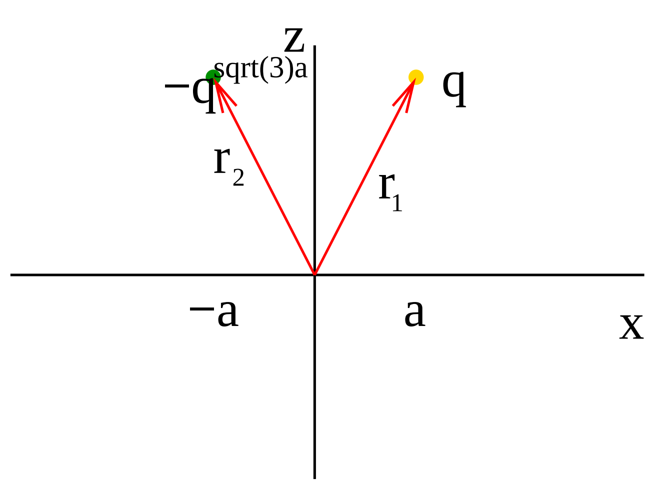
For $l = 1$ the integral vanishes due to the odd parity of the integrand in Eq.(29).

For $l = 2$ I obtain $a_2 = -\frac{5\pi}{32}$.

d) $\sqrt{1-x^2} = P_1^1(x)$.

e) We see that combining c) and d) we have realized the expansion of an associated Legendre function in terms of Legendre polynomials which agrees with the statement made in part a).

Problem 4:



a) We use the principle of superposition to write the total potential as the sum of the potential of the individual charges q_i located at $\mathbf{r}_i$ given by $\Phi_i(\mathbf{r}) = \frac{q_i}{4\pi\epsilon_0\{\mathbf{r}-\mathbf{r}_i|}$. Then,

$$\Phi_q(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}(\frac{1}{|\mathbf{r}-\mathbf{r}_1|} - \frac{1}{|\mathbf{r}-\mathbf{r}_2|}) \tag{30}$$

where the vectors $\mathbf{r}_i$ are indicated in the figure.

b) The problem does not have azimuthal symmetry and thus, the potential has to be expanded in terms of the spherical harmonics. Since

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{Y_{l,m}(\theta,\phi)Y_{l,-m}(\theta',\phi')}{(2l+1)}, \tag{31}$$

we can replace Eq.(30) in Eq.(29) using that $\theta_i' = \pi/6$ for both charges and $\phi_1' = 0$ and $\phi_2' = \pi$.We see that both charges are located at a distance $r = 2a$ from the origin. This means that for $r > 2a$ in Eq.(29) $r_< = 2a$ and $r_> = r$ while for $r < 2a$, $r_< = r$ and $r_> = 2a$. Then we obtain:

$$\Phi(r,\theta,\phi) = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{Y_{l,m}(\theta,\phi)}{(2l+1)}[Y_{l,-m}(\frac{\pi}{6},0) - Y_{l,-m}(\frac{\pi}{6},\pi)]. \tag{32}$$

c) For $r = a$ we get:

$$\Phi(a,\theta,\phi) = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{a^l}{2a^{l+1}} \frac{Y_{l,m}(\theta,\phi)}{(2l+1)}[Y_{l,-m}(\frac{\pi}{6},0) - Y_{l,-m}(\frac{\pi}{6},\pi)]. \tag{33}$$

For $r = 4a$ we get:

$$\Phi(4a,\theta,\phi) = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{2a^l}{4a^{l+1}} \frac{Y_{l,m}(\theta,\phi)}{(2l+1)}[Y_{l,-m}(\frac{\pi}{6},0) - Y_{l,-m}(\frac{\pi}{6},\pi)]. \tag{34}$$

d) Outside the shell the potential is given by

$$\Phi(r,\theta,\phi) = \Phi_q + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{B_{l,m}}{r^{l+1}} Y_{l,m}(\theta,\phi). \tag{35}$$

e) At $r = a$ we know that

$$\Phi(a,\theta,\phi) = 0. \tag{36}$$

Bonus: At $r = a$:

$$\frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{a^l}{(2a)^{l+1}} \frac{[Y_{l,-m}(\frac{\pi}{6},0) - Y_{l,-m}(\frac{\pi}{6},\pi)]}{2l+1} Y_{l,m}(\theta,\phi) + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{B_{l,m}}{a^{l+1}} Y_{l,m}(\theta,\phi) = 0. \tag{37}$$

Then,

$$B_{l,m} = \frac{q}{\epsilon_0}\frac{a^l}{2^l}\frac{[Y_{l,-m}(\frac{\pi}{6},\pi) - Y_{l,-m}(\frac{\pi}{6},0)]}{2l+1} \tag{38}$$

Using the explicit form of the spherical harmonics

$$Y_{l,m}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\phi}, \tag{39}$$

we see that $B_{l,m} = 0$ if m is even and for m odd:

$$B_{l,m} = \frac{q}{\epsilon_0}\frac{a^l}{2^{l-1}(2l+1)}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\frac{\sqrt{3}}{2}). \tag{40}$$