Midterm Exam

P571

October 7, 2008

SOLUTION:

**Problem 1**:

a) Lets calculate $\nabla \times \mathbf{F}$. We find:

$$\nabla \times \mathbf{F} = (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\hat{\mathbf{z}}, \tag{1}$$

since $\mathbf{F}$ is independent of $z$. Then, the curl of $\mathbf{F}$ is along the $z$ axis while $\mathbf{F}$ has only components on the $x-y$ plane. This means that

$$\mathbf{F}.(\nabla \times \mathbf{F}) = 0, \tag{2}$$

which means that $\nabla \times \mathbf{F}$ and $\mathbf{F}$ are perpendicular to each other.

b) If we call $\nabla \times \mathbf{F} = \mathbf{V} = V\hat{\mathbf{z}}$ then

$$\mathbf{F} \times (\nabla \times \mathbf{F}) = VF_y\hat{\mathbf{x}} - VF_x\hat{\mathbf{y}}. \tag{3}$$

This indicates that $\mathbf{F} \times (\nabla \times \mathbf{F})$ is on the $x-y$ plane.

c) If $\mathbf{K} = \mathbf{F} \times (\nabla \times \mathbf{F})$ we have that

$$\mathbf{K} = \mathbf{F} \times \mathbf{V}, \tag{4}$$

from the definition of vector product we know that $\mathbf{K}$ has to be perpendicular to both $\mathbf{F}$ and $\mathbf{V}$; thus the angle that $\mathbf{K}$ makes with $\mathbf{F}$ is $90^o$.

**Problem 2**:

a) In 3D each of the four indices can take three values. This means that $T_{iklm}$ has $3^4 = 81$ components.

b) Since $T_{iklm}$ is antisymmetric with respect to all pairs of indices it means that all its non-zero components have to have all four indices different from each other. But this is impossible in 3D because each index can take three possible values. Thus $T_{iklm} = 0$ and it does not have any independent component.

c) In 4D each of the four indices can take four values. This means that $T_{iklm}$ has $4^4 = 256$ components.

d) Since $T_{iklm}$ is antisymmetric with respect to all pairs of indices it means that all its non-zero components have to have all four indices different from each other. In 4D this means that there are $4 \times 3 \times 2 \times 1 = 24$ components different from zero. Now let's assume that $T_{1234} = A$. Due to the symmetry properties of the tensor all the other 23 non-zero components will have the value $A$ or $-A$ and thus, there is only one independent component.

**Problem 3**:

a) Since there is a point charge at $r = a$ we can divide the space inside the sphere in two regions. Region I for $r < a$ and region II for $r > a$, and solve the Laplace equation in both regions since there is no charge in them.

b) In principle we would have to propose one solution for each region as said in part a) but, since we have studied in class the contribution to the potential due to a point charge, we can use the principle of superposition and propose as solution:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta) + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(cos\theta), \tag{4}$$

where we have used that the problem has azimuthal symmetry, by chosing the $z$ axis along the direction in which the charge $q$ is; the division in two regions is implicitly contained in the expression for the charge's potential where $r_< = r$ and $r_> = a$ in region I and $r_< = a$ and $r_> = r$ in region II. Then, since the contribution due to the spherical shell cannot diverge at $r = 0$ only positive powers of $r$ appear in our expression. Using the principle of superposition we only have one set of undetermined coefficients to find and for this we use the boundary condition:

$$\Phi(r = R, \theta) = 0. \tag{5}$$

c)Using (5) we find that

$$A_l = -\frac{q}{4\pi\epsilon_0}\frac{a^l}{R^{2l+1}}, \tag{6}$$

and replacing (6) in (4) we obtain:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{-a^l}{R^{2l+1}}r^l + \frac{r_<^l}{r_>^{l+1}}\right)P_l(cos\theta). \tag{7}$$

**Problem 4**:

a)

$$\sum_{M=-L}^{L} \psi(r,\theta,\phi)^{*}\psi(r,\theta,\phi) = |f(r)|^{2} \sum_{M=-L}^{L} Y_{L}^{M*}(\theta,\phi)Y_{L}^{M}(\theta,\phi). \tag{8}$$

In order to evaluate the sum over the $Y_L^M$ we can use the theorem of addition of spherical harmonics:

$$P_L(cos\gamma) = \frac{4\pi}{2L+1} \sum_{M=-L}^{L} Y_{L}^{M*}(\theta',\phi')Y_{L}^{M}(\theta,\phi), \tag{9}$$

where $\gamma$ is the angle between the directions $(\theta',\phi')$ and $(\theta,\phi)$. If, as in our case, $(\theta',\phi')$ $=(\theta,\phi)$ it means that $\gamma = 0$ and thus $P_L(cos\gamma) = 1$ for all $L$. Which means that

$$\sum_{M=-L}^{L} Y_{L}^{M*}(\theta,\phi)Y_{L}^{M}(\theta,\phi) = \frac{(2L+1)}{4\pi}. \tag{10}$$

Then replacing (10) in (8) we get:

$$\sum_{M=-L}^{L} \psi(r,\theta,\phi)^{*}\psi(r,\theta,\phi) = \frac{(2L+1)}{4\pi}|f(r)|^{2} \tag{11}$$

b) We see that the expression depends only on $r$.