Midterm Exam

P571

October 1, 2009

SOLUTION:

Problem 1:

a)

$$\mathrm{F}_a{}^b = g_{ac}\mathrm{F}^{cb} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$\mathrm{F}^a{}_b = g_{bc}\mathrm{F}^{ac} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

b)

$$\mathrm{F}_a{}^b\mathrm{F}^a{}_b = \sum_{a=0}^{3}\sum_{b=0}^{3}\mathrm{F}_a{}^b\mathrm{F}^a{}_b = 2(B^2 - E^2).$$

As the double contraction of two tensors $\mathrm{F}_a{}^b\mathrm{F}^a{}_b$ is a tensor. Its rank is 0 because we are contracting all the indices, and we see that the explicit expression is a scalar.

c) $\mathrm{F}_a{}^b\mathrm{F}^c{}_b$ is a tensor of rank 2 of the form $\mathrm{G}_a{}^c$ and

$$\mathrm{F}_1{}^b\mathrm{F}^2{}_b = \mathrm{F}_1{}^0\mathrm{F}^2{}_0 + \mathrm{F}_1{}^3\mathrm{F}^2{}_3 = -E_xE_y - B_yB_x.$$

Problem 2:

a) In tensor notation

$$(\mathbf{A}\times\mathbf{B}).(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B})_i(\mathbf{C}\times\mathbf{D})^i = \epsilon_{ijk}A^jB^k\epsilon^{ilm}C_lD_m =$$

$$\epsilon_{ijk}\epsilon^{ilm}A^jB^kC_lD_m = (\delta_j{}^l\delta_k{}^m - \delta_j{}^m\delta_k{}^l)A^jB^kC_lD_m =$$

$$A^lB^mC_lD_m - A^mB^lC_lD_m = A^lC_lB^mD_m - A^mD_mB^lC_l = (\mathbf{A}.\mathbf{C})(\mathbf{B}.\mathbf{D}) - (\mathbf{A}.\mathbf{D})(\mathbf{B}.\mathbf{C}).$$

b)We get

$$(\mathbf{A}\times\mathbf{B}).(\mathbf{A}\times\mathbf{B}) = (\mathbf{A}.\mathbf{A})(\mathbf{B}.\mathbf{B}) - (\mathbf{A}.\mathbf{B})(\mathbf{B}.\mathbf{A}) = A^2B^2 - (\mathbf{A}.\mathbf{B})^2,$$

which vanishes if **A** and **B** are parallel or antiparallel to each other.

Problem 3:

a) In this case $\rho(\mathbf{r}) = Q\delta(x-2)\delta(y+1)\delta(z-1)$.

b) In this case $\rho(\mathbf{r}) = \frac{Q\delta(r-\sqrt{5})\delta(\phi+0.147\pi)\delta(z-1)}{r}$.

c)In this case we can work in cylindrical coordinates so that $\rho(\mathbf{r}) = \frac{Q\delta(r-R)\delta(z)}{2\pi r}$, or in spherical with $\mathbf{r} = (r, \theta, \phi)$ for which $\rho(\mathbf{r}) = \frac{Q\delta(r-R)\delta(\theta-\pi/2)}{2\pi r^2}$, or in spherical with $\mathbf{r} = (r, \cos\theta, \phi)$ for which $\rho(\mathbf{r}) = \frac{Q\delta(r-R)\delta(\cos\theta)}{2\pi r^2}$.

Problem 4:

a) We see that

$$2\hat{z} - 4\hat{y} = \hat{x} \times \mathbf{B} = B_y\hat{z} - B_z\hat{y}, \tag{1}$$

$$4\hat{x} - \hat{z} = \hat{y} \times \mathbf{B} = B_z\hat{x} - B_x\hat{z}, \tag{2}$$

$$\hat{y} - 2\hat{x} = \hat{z} \times \mathbf{B} = B_x\hat{y} - B_y\hat{x}. \tag{3}$$

Then, we see that $B_x = 1$, $B_y = 2$, and $B_z = 4$, i.e.,

$$\mathbf{B} = (1, 2, 4).$$

b) In tensor form

$$\nabla.(\mathbf{u} \times \mathbf{v}) = \partial_a \epsilon^{abc} u_b v_c = \epsilon^{abc} v_c \partial_a u_b + \epsilon^{abc} u_b \partial_a v_c =$$

$$\epsilon^{abc} v_c \partial_a u_b + \epsilon^{bca} u_b \partial_a v_c = v_c \epsilon^{abc} \partial_a u_b - u_b \epsilon^{bac} \partial_a v_c = \mathbf{v}.(\nabla \times \mathbf{u}) - \mathbf{u}.(\nabla \times \mathbf{v}) = 0.$$