Midterm Exam #2

P571

November 5, 2009

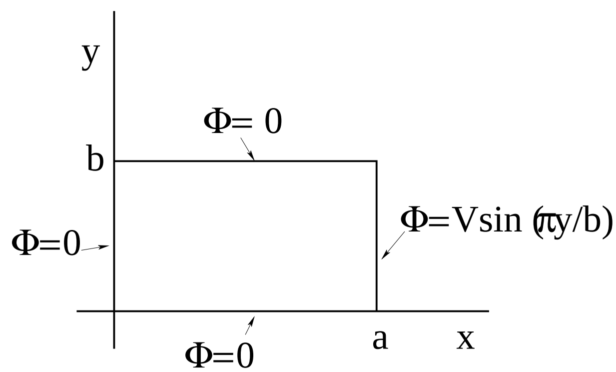
SOLUTION:

Problem 1:

a) We have to solve Laplace's equation because the region is free of charge.

b) The equation is

$$\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 0. \tag{1}$$



c) A general solution will have the form:

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}, \tag{2}$$

It satisfies Eq.(1) and the b.c.'s for $\Phi = 0$. The coeficient A_n is determined from the b.c. at $x = a$:

$$V \sin\frac{\pi y}{b} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi a}{b}, \tag{3}$$

from orthogonality of the sines we see that $A_n = 0$ for all $n \neq 1$ and

$$A_1 = \frac{V}{\sinh\frac{\pi a}{b}}. \tag{4}$$

Then,

$$\Phi(x,y) = \frac{V}{\sinh\frac{\pi a}{b}} \sin\frac{\pi y}{b} \sinh\frac{\pi x}{b}. \tag{5}$$

d) At the center of the region

$$\Phi(\frac{a}{2},\frac{b}{2}) = \frac{V}{\sinh\frac{\pi a}{b}} \sinh\frac{\pi a}{2b}. \tag{6}$$

Problem 2:

a) We need to solve Laplace's equation in two different regions defined by the charged spherical surface. We cannot work in a single region because there is charge at $r = a$ and Laplace's equation is not valid there.

b) I expect to obtain the solution in terms of powers of r and Legendre polynomials because the boundary conditions are defined on a sphere and there is azimuthal symmetry.

c) In region I ($r \leq a$) I propose:

$$\Phi^I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{6}$$

we set to zero the coefficient of negative powers of r since the potential cannot diverge at $r = 0$. In region II ($r \geq a$) I propose:

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), \tag{7}$$

where the coefficients of positive powers of r have been set to zero because the potential has to vanish as $r \to \infty$.

d) In order to determine the two sets of undetermined coefficients A_l and B_l I need two boundary conditions. We know that at $r = a$ the potential has to be continuous then:

$$\Phi^I|_{r=a} = \Phi^{II}|_{r=a}. \tag{8}$$

We also know that the normal component of the electric field across a charged surface has a jump equal to σ/ϵ_0 where σ is the surface density of charge. In this case the normal to the surface is the radial component, then $E_n = E_r = -\frac{\partial \Phi}{\partial r}$ and the second boundary condition becomes:

$$\frac{\partial \Phi^{II}}{\partial r}|_{r=a} - \frac{\partial \Phi^{I}}{\partial r}|_{r=a} = -\frac{\sigma_0 \cos\theta}{\epsilon_0}. \tag{9}$$

d) From Eq.(8) we find that

$$A_l = \frac{B_l}{a^{2l+1}}. \tag{10}$$

And from Eq.(9) we obtain:

$$\sum_{l=0}^{\infty}[-(l+1)\frac{B_l}{a^{l+2}} - lA_l a^{l-1}]P_l(\cos\theta) = -\frac{\sigma_0 \cos\theta}{\epsilon_0}. \tag{11}$$

Notice that $\cos\theta = P_1(\cos\theta)$. Thus multiplying both sides of Eq.(11) by $P_m(\cos\theta)$ and integrating over $\cos\theta$ in the interval $[-1, 1]$ we obtain:

$$-(m+1)\frac{B_m}{a^{m+2}} - mA_m a^{m-1} = -\frac{\sigma_0}{\epsilon_0}\delta_{m,1}. \tag{12}$$

Replacing Eq.(10) in Eq.(12) we get:

$$(2m+1)\frac{B_m}{a^{m+2}} = \frac{\sigma_0}{\epsilon_0}\delta_{m,1}. \tag{13}$$

Then, if $m \neq 1$, $A_m = B_m = 0$. If $m = 1$

$$B_1 = \frac{\sigma_0 a^3}{3\epsilon_0}, \tag{14}$$

and

$$A_1 = \frac{\sigma_0}{3\epsilon_0}. \tag{15}$$

Replacing in Eq.(6) and (7) we obtain:

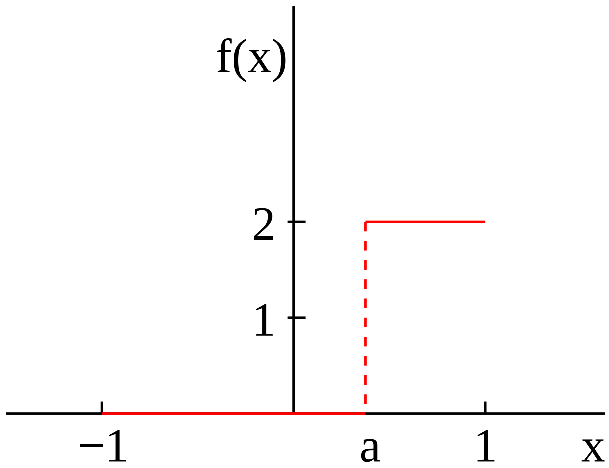
$$\Phi^I(r,\theta) = \frac{\sigma_0}{3\epsilon_0} r\cos\theta, \tag{16}$$

and

$$\Phi^{II}(r,\theta) = \frac{\sigma_0 a^3}{3\epsilon_0 r^2}\cos\theta. \tag{17}$$

Problem 3:

a) We can expand $f(x)$ in terms of Legendre polynomials because they form a set of orthogonal functions in the interval $[-1, 1]$ in which $f(x)$ is defined.



A formal expression for the expansion is given by

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x). \tag{18}$$

b) Using orthogonality of the Legendre polynimials we find that

$$a_l = (2l+1) \int_a^1 P_l(x) dx. \tag{19}$$

To find the first 3 coefficients we need to set $l = 0,1,$ and 2 and perform the integral. We obtain:

$$a_0 = (1 - a), \tag{20}$$

$$a_1 = \frac{3}{2}(1 - a^2), \tag{21}$$

$$a_2 = \frac{5a}{2}(1 - a^2). \tag{22}$$

c) Using the hint we can easily solve the integral in Eq.(19) and we obtain:

$$a_l = (2l+1)\int_a^1 P_l(x)dx = (P_{l+1}(x) - P_{l-1}(x))|_a^1 = P_{l-1}(a) - P_{l+1}(a), \tag{23}$$

where we have used that $P_l(\pm 1) = 1$. Then we obtain that

$$a_1 = P_0(a) - P_2(a) = \frac{3}{2}(1-a^2), \tag{24}$$

and

$$a_2 = P_1(a) - P_3(a) = \frac{5a}{2}(1-a^2). \tag{25}$$

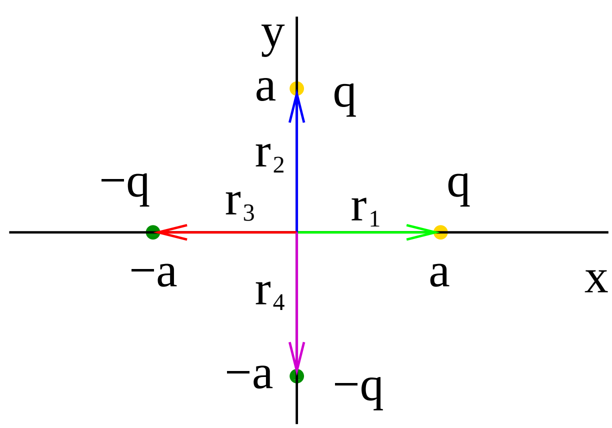
d) Now let's calculate

$$\int_{-1}^1 [f(x)]^2 dx = \int_{-1}^1 \sum_{l,m} a_l a_m P_l(x) P_m(x) dx = \sum_{l,m} a_l a_m \frac{2}{2l+1}\delta_{l,m} = 2\sum_{l=0}^{\infty} \frac{a_l^2}{2l+1}. \tag{26}$$

Using the result obtained in Eq.(23) valid for $l > 0$ and Eq.(20) we find that

$$\int_{-1}^1 [f(x)]^2 dx = 2(1-a)^2 + \sum_{l=1}^{\infty} \frac{(P_{l-1}(a) - P_{l+1}(a))^2}{2l+1}. \tag{27}$$

Problem 4:



a) We use the principle of superposition to write the total potential as the sum of the potential of the individual charges q_i located at $\mathbf{r}_i$ given by $\Phi_i(\mathbf{r}) = \frac{q_i}{4\pi\epsilon_0\{\mathbf{r}-\mathbf{r}_i|}$. Then,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}(\frac{1}{|\mathbf{r}-\mathbf{r}_1|} + \frac{1}{|\mathbf{r}-\mathbf{r}_2|} - \frac{1}{|\mathbf{r}-\mathbf{r}_3|} - \frac{1}{|\mathbf{r}-\mathbf{r}_4|}) \tag{28}$$

where the vectors $\mathbf{r}_i$ are indicated in the figure.

b) The problem does not have azimuthal symmetry and thus, the potential has to be expanded in terms of the spherical harmonics. Since

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{(2l+1)r_>^{l+1}} Y_{l,m}(\theta,\phi) Y_{l,-m}(\theta',\phi'), \tag{29}$$

we can replace Eq.(29) in Eq.(28) using that $\theta_i = \pi/2$ for all the charges and $\phi_1 = 0$, $\phi_2 = \pi/2$, $\phi_3 = \pi$, and $\phi_4 = 3\pi/2$. Since for $r > a$, $r_< = a$ and $r_> = r$ we obtain,

$$\Phi(r,\theta,\phi) = \frac{q}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a^l}{(2l+1)r^{l+1}} Y_{l,m}(\theta,\phi)[Y_{l,-m}(\frac{\pi}{2},0) + Y_{l,-m}(\frac{\pi}{2},\frac{\pi}{2}) - Y_{l,-m}(\frac{\pi}{2},\pi) - Y_{l,-m}(\frac{\pi}{2},\frac{3\pi}{2})]. \tag{30}$$