Midterm Exam

P571

September 27, 2011

SHOW ALL WORK TO GET FULL CREDIT!

PART I:**ONLY TWO OF THE FOUR PROBLEMS WILL BE GRADED**. Take a look at the 4 problems. Each of them is worth 25 points. To make sure that you have enough time to do your work you will have to turn in only 2 of the 4 problems. **If you turn more than 2 problems only the two on top will be graded and 5 points will be deducted from your grade.**

PART II: Take the test home and bring **ALL** the problems solved on Tuesday October 4. Your grade for the test will be the **sum of the two** parts. A perfect score is worth 150 points.

Problem 1: Consider a cartesian frame of reference S with perpendicular axis x_1 and x_2 and a frame S' with oblique axis x'_1 which is parallel to x_1 and x'_2 which is parralel to the vector $\mathbf{r} = (1, 2)$ in frame S.

a) Find the angle α between x'_1 and x'_2.(5 points)

b) Find the metric tensor g_{ij} in S'.(5 points)

c) Find g^{ij} in S'.(5 points)

d) Consider the vector $r'^i = (1, 1)$ in system S'. Express its magnitude in terms of a tensor contraction in frame S' and provide its value $|\mathbf{r}'|$.(5 points)

e) Construct the tensor $T'^{ij} = r'^i r'^j$ and provide $T'^i{}_j$ and T'_{ij}. (5 points)

Problem 2:

a) Write and prove in tensor notation (justifying all your steps) the identity given by

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A}.(\mathbf{B} \times \mathbf{D})]\mathbf{C} - [\mathbf{A}.(\mathbf{B} \times \mathbf{C})]\mathbf{D}$$

(10 points)

b) Knowing that **A**, **B**, **C**,and **D** are vectors is $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$ a tensor? Provide its rank and explain.(5 points)

c) Is $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$ a tensor or a pseudotensor? Why? (5 points)

d) What is the value of $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$ if **A**, **B**, **C**, and **D** are all coplanar (in the same plane)? Explain. (5 points)

Problem 3:

a) Find the electrical potential $\Phi(x, y)$ inside the region defined by $0 \leq x \leq \infty$ and $0 \leq y \leq a$ if the potential at the region boundaries is given by $\Phi(x, 0) = \Phi(x, a) = 0$, $\lim_{x\to\infty}\Phi(x, y) = 0$, and $\Phi(0, y) = V$, where V is a constant. (10 points)

b) Now a charge q is placed inside the volume at point $(x_0, y_0) = (a, a/2)$. Find the electrical potential $\Phi(x, y)$ inside the region defined by $0 \leq x \leq \infty$ and $0 \leq y \leq a$ if the potential at the region boundaries is given by $\Phi(x, 0) = \Phi(x, a) = 0$, $\lim_{x\to\infty}\Phi(x, y) = 0$, and $\Phi(0, y) = V$, where V is a constant. Hint: Use the superposition principle.(15 points)

Problem 4:

Consider the tensor $\mathrm{G}^{\alpha\rho}$, where α and ρ take the values 0, 1, 2, and 3 with $x_0 = ct$ (c is a constant), $x_1 = x$, $x_2 = y$, and $x_3 = z$ in frame of reference S. In this frame

$$\mathrm{G}^{\alpha\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -H_z & H_y \\ 0 & H_z & 0 & -H_x \\ 0 & -H_y & H_x & 0 \end{pmatrix}. \tag{1}$$

The metric tensor in frame S is

$$g_{\alpha\rho} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

a) Wrire an expression for $G_{\mu,\nu}$. (5 points)
b) Consider the contraction $G_{\mu}{}^{\mu}$. What is the rank of the tensor $G_{i}{}^{i}$? Provide its value in terms of H_i.(5 points)
c) What is the rank of $G_{\mu,\nu}G^{\sigma\rho}$? Provide the value of the component with $\mu = 1$, $\nu = 3$, $\sigma = 1$, and $\rho = 2$. (5 points)
d) What is the rank of $G_{\mu,\nu}G^{\nu\rho}$? Provide the value of the component with $\mu = 1$ and $\rho = 2$. (5 points)
e) Calculate the dual tensor of $G^{\mu,\nu}$.(5 points)