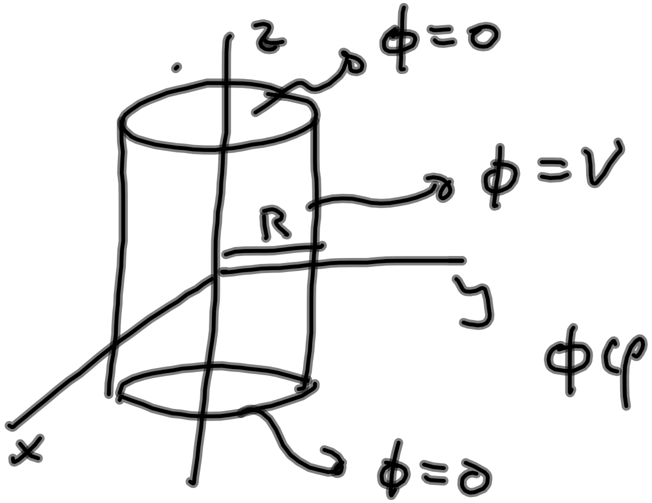
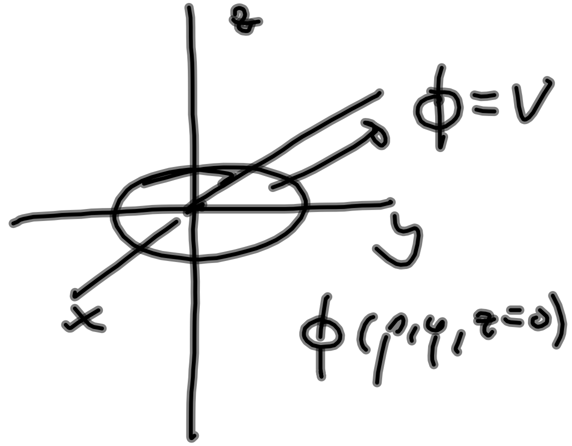
10/25

Laplace's eq. in cylindrical coordinates



$\phi(\rho=R)=V$

or



$\phi(\rho,\varphi,z=0)=V$

In this case we will use (ρ,φ,z) as coordinates

Oct 25-2:06 PM

Then we have to write

$\nabla^2 \phi(\rho, \varphi, z) = 0$ and solve it.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (1)$$

Separation of variables to solve (1):

We propose:

$$\phi(\rho, \varphi, z) = P(\rho)\,Q(\varphi)\,Z(z) \qquad (2)$$

Plug (2) in (1):

Oct 25-2:13 PM

$$QZ\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right)+\frac{PZ}{\rho^2}\frac{\partial^2 Q}{d\varphi^2}+PQ\frac{\partial^2 Z}{\partial z^2}=0$$

Divide by PQZ:

$$\underbrace{\frac{1}{P}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right)+\frac{1}{\rho^2 Q}\frac{\partial^2 Q}{\partial\varphi^2}}_{-k^2}+\underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{+k^2}=0$$

(In not $\phi = V(\rho,\varphi)$ at top or bottom of cylinder)

Then

$$\frac{\partial^2 Z}{\partial z^2}=k^2 Z \Rightarrow \boxed{Z(z)\propto e^{\pm kz}}$$

Oct 25-2:16 PM

Now we have:

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\rho^2 Q}\frac{\partial^2 Q}{\partial\varphi^2}=-k^2 \qquad (2)$$

Multiply (2) by ρ^2 :

$$\frac{\rho^2}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\underbrace{\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2}}_{-\nu^2}=-k^2\rho^2$$

and then

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+k^2\rho^2=\nu^2 \qquad (3)$$

Then

$$\boxed{Q(\varphi)\propto e^{\pm i\nu\varphi}}$$

periodic dependence on φ.

Oct 25-2:21 PM

Now we need to solve ③:

Multiply ③ by P:

$$\rho \frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + (k^2\rho^2 - \nu^2) P = 0$$

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)P = 0$$

If $x = k\rho \Rightarrow dx = k\,d\rho$ then we obtain

$$k^2 \frac{d^2P}{dx^2} + \frac{k^2}{x}\frac{dP}{dx} + \left(k^2 - \frac{\nu^2 k^2}{x^2}\right)P = 0$$

Oct 25-2:26 PM

Divide by k^2:

$$\boxed{\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)P = 0}$$

Bessel's equation

To solve for P we are going to find two solutions P_1 and P_2 so that

$$P(x) = c_1 P_1(x) + c_2 P_2(x)$$

Oct 25-2:29 PM

Frobenius' method.

- Useful to solve linear, second order, homogeneous ordinary differential eqs. of the form:

$$y'' + P(x)\,y' + Q(x)\,y = 0 \qquad (1)$$

with $y'' = \dfrac{d^2y}{dx^2}$ $\quad y' = \dfrac{dy}{dx}$

$P(x)$ and $Q(x)$ are functions of x.

$P(x) = \frac{1}{x}$ and $Q(x) = 1 - \frac{\nu^2}{x^2}$ for Bessel's eq.

Oct 25-2:33 PM

We propose a solution of the form:

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (2)$$

and set up equations to solve for a_λ and k.

Example: Bessel's equation.

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (3)$$

From (2):

$$y'(x) = \sum_{\lambda=0}^{\infty} (k+\lambda) a_\lambda x^{k+\lambda-1} \qquad (4)$$

$$y''(x) = \sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda-2} \qquad (5)$$

Oct 25-2:35 PM

Plug ②, ④, and ⑤ in ③:

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda} + \sum_{\lambda=0}^{\infty} (k+\lambda) x^{k+\lambda} a_\lambda +$$

$$+ \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} - \sum_{\lambda=0}^{\infty} a_\lambda n^2 x^{k+\lambda} = 0$$

For $\lambda = 0$ we obtain the coefficients for the lowest power of x which is x^k:

$$a_0 k(k-1) x^k + k x^k a_0 - a_0 n^2 x^k = 0$$

then we request

$$a_0 k(k-1) + k a_0 - a_0 n^2 = 0 \quad \text{indicial equation}$$

$$a_0 k^2 - a_0 n^2 = 0 \Rightarrow \boxed{k = \pm n}$$ since a_0 does not have to be zero.

The two values of k, in principle, should give us the two independent solutions $y_1(x)$ and $y_2(x)$.

Now consider $\lambda = 1$. The lowest power of x is x^{k+1}. We need to make sure that the coefficient of x^{k+1} vanishes.

$$a_1 \left[(k+1)k + k + 1 - n^2 \right] = 0 \qquad 4$$

or

$$a_1 (k+1-n)(k+1+n) = 0 \qquad \text{if } k = \pm n$$

We see that this is satisfied if $\boxed{a_1 = 0}$.

Exception: for $n = -\frac{1}{2}$ 4 vanishes even if $a_1 \neq 0$.

Oct 25-2:49 PM

Note: if one finds k that solves (9) with a_1 arbitrary then we need to take $a_0 = 0$ and the same solutions are going to be found.

Now if $a_0 \neq 0$ and $a_1 = 0$ let's look at $\lambda = j$: in this case the lowest power of x that appears is x^{k+j}. We request that its coefficient vanishes:

$$a_j\left[(m+j)(m+j-1) + (m+j) - n^2\right] + a_{j-2} = 0 \qquad (6)$$

You see that for $a_0 \neq 0$ $a_2, a_4, \dots$ can be obtained and since $a_1 = 0 \Rightarrow a_3 = a_5 = \dots a_{2n+1} = 0$

In ⑥ rename j so that $j \equiv j-2$

$$a_{j+2}\left[(m+j+2)(m+j+1)+(m+j+2)-n^2\right]+a_j=0$$

then

$$a_{j+2}=-\frac{a_j}{(j+2)(2n+j+2)}$$ ⊛ recurrence relation

Then:

$$a_2=\frac{-a_0}{2(2n+2)}=-\frac{a_0\, n!}{2^2\, 1!\,(n+1)!}$$

$$a_4=\frac{-a_2}{4(2n+4)}=\frac{a_0\, n!}{2^4\, 2!\,(n+2)!}$$

Oct 25-3:01 PM

in general:

$$a_{2p} = \frac{(-1)^p a_0 n!}{2^{2p} p! (n+p)!} \quad \text{and} \quad a_{2p+1} = 0 \quad \text{for all } p.$$

Then using the expressions for a_{2p} we obtain: $k=n$

$$y_1(x) = a_0 x^n \left[1 - \frac{n! x^2}{2^2 1! (n+1)!} + \frac{n! x^4}{2^4 2! (n+2)!} + \dots \right]$$

$$= a_0 2^n n! \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j}{j! (n+j)!} \left(\frac{x}{2}\right)^{n+2j}}_{J_n(x)}$$

Bessel's function of order n. [Ch. 11].

Oct 25-3:06 PM

Taking $k = -n$ you get $y_2(x)$.

Oct 25-3:11 PM

Notice that in ⊛

$$a_{j+2} = \frac{-a_j}{(j+2)(2n+j+2)}$$

if $j+2 = -2n$ or $j = -2(n+1)$ a_{j+2} diverges.

Then $J_{-n}(x)$ cannot be found by setting $k=-n$.

In this case it can be found that

$J_{-n}(x) = (-1)^n J_n(x)$ [in Ch. 11].

Then Frobenius method is very useful but sometimes it does not work. You need to check that the a_j's you find do not diverge.

Oct 25-3:13 PM

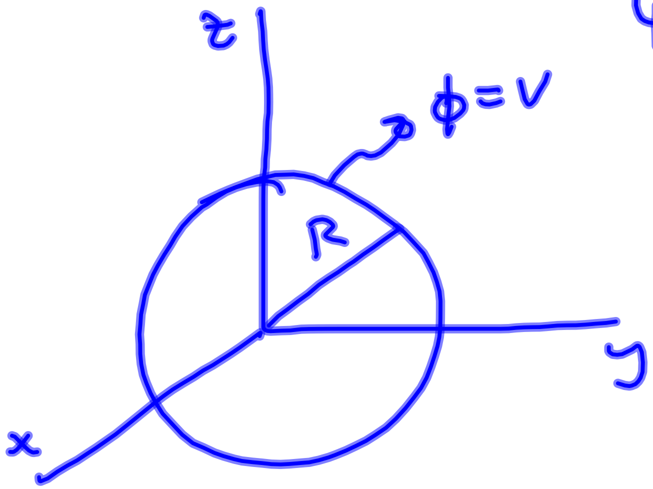
Then

$$\phi(\rho, \varphi, z) \propto J_{\pm n}(k\rho)\, e^{\pm i\nu\varphi}\, e^{\pm kz}$$

Oct 25-3:17 PM

Laplace's equation is spherical coordinates.

$$\phi(r=R, \varphi, \theta) = V(\theta, \varphi)$$



We should use spherical coordinates to solve

$$\nabla^2 \phi = 0.$$

proposal

$$\phi = \phi(r, \theta, \varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi)$$

Oct 25-3:20 PM