Midterm Exam II

P571
November 15, 2012

SOLUTION:

Problem 1:

a) We need to write in tensor form $\hat{n} \times \mathbf{a} = \mathbf{B}$.

$$\epsilon_{ijk} n^j a^k = B_i. \tag{1}$$

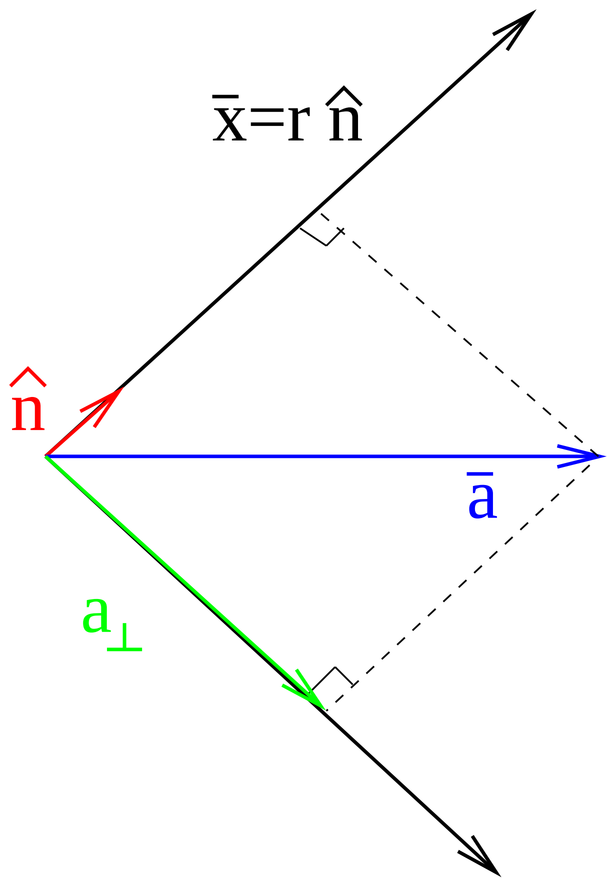
B_i is a pseudotensor of rank 1 because $\hat{n}$ and $\mathbf{a}$ are vectors and ϵ_{ijk} is a pseudotensor.

b) Show that $(\hat{n} \times \mathbf{a}) \times \hat{n} = \mathbf{a} - (\mathbf{a}.\hat{n})\hat{n}$ using tensor notation:

$$\epsilon^{rst} B_s n_t = \epsilon^{rst} \epsilon_{suv} n^u a^v n_t = \epsilon^{str} B_s n_t = \epsilon^{rst} \epsilon_{suv} n^u a^v n_t == (\delta^t{}_u \delta^r{}_v - \delta^t{}_v \delta^r{}_u) n^u a^v n_t = n^t n_t a^r - a^t n_t n^r = a^r - a^t n_t n^r, \tag{2}$$

which is $\mathbf{a} - (\mathbf{a}.\hat{n})\hat{n}$.

c) Eq.(2) represents the component of $\mathbf{a}$ along the direction perpendicular to $\hat{n}$ and it is shown in the figure below.



d) Let's calculate $\nabla.\hat{n}$ in tensor notation:

$$\delta_i[\frac{x^i}{(x_k x^k)^{1/2}}] = \frac{\partial_i x^i (x_k x^k)^{1/2} - x^i \partial_i (x_k x^k)^{1/2}}{x_k x^k} =$$

$$\frac{3r}{r^2} - \frac{x^i}{r^2}\frac{1}{2r}(\partial_i x_k x^k + x_k \partial_i x^k) = \frac{3}{r} - \frac{x^i}{2r^3}(\partial_i g_{kl} x^l x^k + x_k \delta_i{}^k) =$$

$$\frac{3}{r} - \frac{x^i}{2r^3}(\delta_i{}^l g_{kl} x^k + x_i) = \frac{3}{r} - \frac{x^i}{2r^3} 2x_i = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}. \quad (3)$$

e) Now we need to calculate $r(\mathbf{a}.\nabla)\hat{n}$ using tensor notation:

$$ra^i \partial_i \frac{x^j}{(x_k x^k)^{1/2}} = ra^i[\frac{\partial_i x^j r}{r^2} - \frac{x^j \partial_i (x_k x^k)}{2r^3}] =$$

$$ra^i[\frac{\delta_i{}^j}{r} - \frac{x^j}{2r^3}(\partial_i x_k x^k + x_k \partial_i x^k)] =$$

$$ra^i[\frac{\delta_i{}^j}{r} - \frac{x^j}{2r^3}(\partial_i g_{kl} x^l x^k + x_k \delta_i{}^k)] =$$

$$ra^i[\frac{\delta_i{}^j}{r} - \frac{x^j}{2r^3}(g_{kl}\delta_i{}^l x^k + x_i)] =$$

$$ra^i[\frac{\delta_i{}^j}{r} - \frac{x^j}{2r^3}(x_i + x_i)] =$$

$$a^j - a^i x_i \frac{x^j}{r^2} = a^j - a^i n_i n^j. \quad (4)$$

Problem 2:

a) It is a tensor of rank 0 because it arises from the contraction of two tensors of rank 1: X^α and U^α.

b)

$$X_\alpha X^\alpha = g_{\alpha\beta} X^\beta X^\alpha = X^0 X^0 - X^1 X^1 - X^2 X^2 - X^3 X^3 = 1 - 1 - 0 - 1/4 = -1/4.$$

$$U_\alpha X^\alpha = g_{\alpha\beta} U^\beta X^\alpha = U^0 X^0 - U^1 X^1 - U^2 X^2 - U^3 X^3 = c(1/2 - \sqrt{3}/2 - 0 - 1/2 = -\frac{\sqrt{3}}{2}c. \quad (5)$$

c) Let's calculate the denominator at the values of X and U provided::

$$[\frac{1}{c^2}(U_\alpha X^\alpha)^2 - X_\alpha X^\alpha]^{3/2} = [\frac{1}{c^2} c^2 \frac{3}{4} + \frac{1}{4}]^{3/2} = 1. \quad (6)$$

Then

$$E_x = -F^{01} = -\frac{q}{c}(X^0U^1 - X^1U^0) = \frac{q}{2}(1-\sqrt{3}), \tag{7}$$

$$E_y = -F^{02} = -\frac{q}{c}(X^0U^2 - X^2U^0) = -q, \tag{8}$$

$$E_z = -F^{03} = -\frac{q}{c}(X^0U^3 - X^3U^0) = \frac{-3q}{4}, \tag{9}$$

$$B_x = -F^{23} = -\frac{q}{c}(X^2U^3 - X^3U^2) = \frac{q}{2}, \tag{10}$$

$$B_y = F^{13} = \frac{q}{c}(X^1U^3 - X^3U^1) = \frac{q}{4}(4-\sqrt{3}), \tag{11}$$

$$B_z = -F^{12} = -\frac{q}{c}(X^1U^2 - X^2U^1) = -q. \tag{12}$$

d)

$$F^{\alpha\beta}F_{\alpha\beta} = 2(B^2 - E^2) = -\frac{q^2}{4}. \tag{13}$$

e)

$$X'^{\alpha} = M^{\alpha}{}_{\beta}X^{\beta} = (\gamma - \beta\gamma, -\beta\gamma + \gamma, 0, 1/2) = (0.71, 0.71, 0, 1/2). \tag{14}$$

$$U'^{\alpha} = M^{\alpha}{}_{\beta}U^{\beta} = c(\frac{\gamma}{2} - \beta\gamma\frac{\sqrt{3}}{2}, -\frac{\beta\gamma}{2} + \frac{\gamma\sqrt{3}}{2}, 1, 1) = c(0.224, 0.7413, 1, 1). \tag{15}$$

f)

$$E'_y = -F'^{02} = -\frac{q}{c}(X'^0U'^2 - X'^2U'^0) = -0.71q, \tag{16}$$

we see that $E'^y \neq E^y$ since it transforms like the component of a tensor of rank 2.

g) No. This result will be the same since $F^{\alpha\beta}F_{\alpha\beta}$ is a tensor of rank 0 and thus it is the same in S and S'.

Problem 3:

a) We need to solve Laplace's equation: $\nabla^2\Phi = 0$.

b) We need to work in spherical coordinates since the boundary consitions are given on spherical surfaces. The potential will be given in terms of the Legendre polynomials $P_l(\cos\theta)$ since the problem has azimuthal symmetry and in terms of powers of r.

c) I propose

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta). \tag{17}$$

We obtain A_l and B_l from the b.c. at $r = a$ and $r = b$. At $r = b$ the potential is $2V_0$ then,

$$2V_0 P_0(\cos\theta) = \sum_{l=0}^{\infty}(A_l b^l + \frac{B_l}{b^{l+1}})P_l(\cos\theta), \tag{18}$$

where we have used that $1 = P_0(x)$. Since Eq.(18) has to hold for each value of l separately we obtain that for $l = 0$

$$A_0 = 2V_0 - \frac{B_0}{b}, \tag{19}$$

while for $l > 0$

$$A_l = -\frac{B_l}{b^{2l+1}}. \tag{20}$$

At $r = a$ the potential is $V_0 \cos\theta = V_0 P_1(\cos\theta)$ then

$$V_0 P_1(\cos\theta) = \sum_{l=0}^{\infty}(A_l a^l + \frac{B_l}{a^{l+1}})P_l(\cos\theta). \tag{21}$$

For $l = 0$ we obtain that

$$A_0 + \frac{B_0}{a} = 0. \tag{22}$$

Plugging Eq.(19) in Eq.(22) and solving for B_0 we obtain:

$$B_0 = \frac{2V_0 ab}{a-b}, \tag{23}$$

and plugging Eq.(23) in Eq.(19) we obtain:

$$A_0 = \frac{-2V_0 b}{a-b}. \tag{24}$$

For $l = 1$ we obtain that plugging Eq.(20) in Eq.(21):

$$-\frac{B_1 a}{b^3} + \frac{B_1}{a^2} = V_0. \tag{25}$$

Then,

$$B_1 = \frac{V_0 a^2 b^3}{b^3 - a^3}, \tag{26}$$

and plugging Eq.(26) in Eq.(20) we obtain:

$$A_1 = \frac{-V_0 a^2}{b^3 - a^3}. \tag{27}$$

For $l > 1$ we obtain that plugging Eq.(20) in Eq.(21):

$$-\frac{B_l a^l}{b^{2l+1}} + \frac{B_l}{a^{l+1}} = 0. \tag{28}$$

Then,

$$B_l = 0, \tag{29}$$

and plugging Eq.(29) in Eq.(20) we obtain:

$$A_l = 0. \tag{30}$$

Then,

$$\Phi(r,\theta) = -\frac{2V_0 b}{a-b} + \frac{2V_0 ab}{(a-b)r} - \frac{V_0 a^2 r}{b^3 - a^3}\cos\theta + \frac{V_0 a^2 b^3}{(b^3-a^3)r^2}\cos\theta. \tag{31}$$

We see that when $a \to 0$, $\Phi(r,\theta) = -2V_0$ which is the potential inside the shell of radius b at a uniform potential.

d) For $r > b$ we propose

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos\theta). \tag{32}$$

At $r = b$ $\Phi = 2V_0$ this means that

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{C_l}{b^{l+1}} P_l(\cos\theta) = 2V_0 P_0(\cos\theta). \tag{33}$$

Then,

$$C_0 = 2V_0 b, \tag{34}$$

and $C_l = 0$ for $l > 0$.Then,

$$\Phi(r,\theta) = \frac{2V_0 b}{r}. \tag{35}$$

e) To find the surface charge density $\sigma(\theta)$ at $r = b$ we know that

$$-\frac{\partial \Phi^{II}}{\partial r}|_{r=b} + \frac{\partial \Phi^{I}}{\partial r}|_{r=b} = \frac{\sigma(\theta)}{\epsilon_0}, \tag{36}$$

where Φ^{II} (Φ^{I}) is the potential for $r > b$ ($r < b$). Then performing the derivatives of Eqs.(35) and (31) we obtain:

$$\sigma(\theta) = \epsilon_0 V_0 [\frac{2}{b-a} - \frac{3a^2\cos\theta}{(b^3-a^3)}]. \tag{37}$$