Homework #11

Problem 1:

Since the potential on the surfaces is given we need to use the Green function for Dirichlet boundary conditions that was obtained in class:

$$G_D(x,y,x',y') = 4\sum_{n=1}^{\infty} \frac{e^{-\frac{\pi n}{a}(x_> - x_<)}}{n} \sin\frac{n\pi y}{a} \sin\frac{n\pi y'}{a}, \tag{1}$$

where $x_>(x_<)$ is the larger (smaller) between x and x'.

In the problem we are considering the density of charge is

$$\rho(x',y') = q\delta(x')\delta(y'-d). \tag{2}$$

The potential inside the volume defined by $-\infty \leq x \leq \infty$ and $0 \leq y \leq a$ is given by:

$$\Phi(x,y) = \frac{1}{4\pi\epsilon_0}\int_V G(x,x',y,y')\rho(x',y')dx'dy' - \frac{1}{4\pi}\oint_S \Phi_s \frac{\partial G_D}{\partial n'} dS'. \tag{3}$$

Since q is at $x' = 0$, I expect to have two different expressions for the potential, one for $x \leq 0$ and another for $x \geq 0$.

Let us first calculate the volume integral which would give us the potential for the charge if all the surfaces were grounded. Because of Eq.(2) we obtain for $x \leq 0$:

$$\Phi_V^I(x,y) = \frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty} \frac{e^{\frac{\pi n x}{a}}}{n} \sin\frac{n\pi y}{a} \sin\frac{n\pi d}{a}, \tag{4}$$

and for $x \geq 0$:

$$\Phi_V^{II}(x,y) = \frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty} \frac{e^{-\frac{\pi n x}{a}}}{n} \sin\frac{n\pi y}{a} \sin\frac{n\pi d}{a}. \tag{5}$$

The two expressions can be combined as

$$\Phi_V(x,y) = \frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty} \frac{e^{-\frac{\pi n}{a}(x_> - x_<)}}{n} \sin\frac{n\pi y}{a} \sin\frac{n\pi d}{a}, \tag{6}$$

where $x_>(x_<)$ is the larger (smaller) between x and 0.

Now let's calculate the surface integral that would give the potential due to the surface potentials in the absence of charges. Notice that this potential only depends on y and it is trivially given by $\Phi(x,y) = \frac{Vy}{a}$. Using the Green function we should obtain the same result, but it will be expanded in terms of $\sin\frac{n\pi y}{a}$.

The surface integral is given by:

$$-\frac{1}{4\pi}\oint_S \Phi_s \frac{\partial G_D}{\partial n'} dS'. \tag{7}$$

In this case the "surface" is the line parallel to the x-axis at $y = a$. The normal is $\hat{n}' = \hat{y}'$. Thus, we have to take the derivative with respect to y' and evaluate it at $y' = a$.

Then,

$$\frac{\partial G_D}{\partial n'}|_S = \frac{\partial G_D}{\partial y'}|_{y'=a} = 4\pi\sum_{n=1}^{\infty} \frac{e^{-\frac{\pi n}{a}(x_> - x_<)}}{a} \sin\frac{n\pi y}{a}(-1)^n, \tag{8}$$

where we have used that $\cos(n\pi y'/a)|_{y'=a} = (-1)^n$.

Then the surface integral becomes:

$$\Phi_s(x,y) = -\frac{V}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi y}{a}\int_{-\infty}^{\infty} e^{-\frac{\pi n}{a}(x_> - x_<)}dx'. \tag{9}$$

Then we need to split the integral in two pieces: for $-\infty \le x' \le x$ we have that $x_< = x'$ and $x_> = x$ while for $x \le x' \le \infty$ we have that $x_< = x$ and $x_> = x'$. Then we have that

$$\int_{-\infty}^{\infty} e^{-\frac{\pi n}{a}(x_> - x_<)}dx' = \int_{-\infty}^{x} e^{-\frac{\pi n}{a}(x-x')}dx' + \int_{x}^{\infty} e^{-\frac{\pi n}{a}(x'-x)}dx' = \frac{2a}{n\pi}. \tag{10}$$

Putting this result in Eq.(9) we obtain:

$$\Phi_s(x,y) = -\frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi y}{a}. \tag{11}$$

Then the total potential is given by the sum of Eq.(6) and Eq.(11):

$$\Phi(x,y) = \frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{e^{-\frac{\pi n}{a}(x_> - x_<)}}{n}\sin\frac{n\pi y}{a}\sin\frac{n\pi d}{a} - \frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi y}{a}. \tag{12}$$

Important detail: notice that this is the same result that you obtained in Problem 6 of Hw#8. The surface term, i.e., Eq.(11), provides the contribution to the potential coming from the plane at potential V which was Vy/a. Notice that the Fourier series $-2\sum_{n=1}^{\infty}(-1)^n\frac{\sin x}{n} = x$ for $-\pi < x < \pi$. This means that replacing $x \to y\pi/a$ we obtain that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi y}{a} = -y\pi/(2a)$ and then the expression in Eq,(11) is equal to Vy/a as expected.