Homework #11

Problem 2:

Since the potential on the surfaces is given we need to use the Green function for Dirichlet boundary conditions that was obtained in class:

$$G_D(x,y,x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\frac{\sinh[(b-y_>)\frac{n\pi}{a}]\sinh(y_<\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})}, \tag{1}$$

where $y_>(y_<)$ is the larger (smaller) between y and y'.

In the problem we are considering the density of charge is

$$\rho(x',y') = q\delta(x'-a/2)\delta(y'-b/2). \tag{2}$$

The potential inside the volume defined by $0 \leq x \leq a$ and $0 \leq y \leq b$ is given by:

$$\Phi(x,y) = \frac{1}{4\pi\epsilon_0}\int_V G(x,x',y,y')\rho(x',y')dx'dy' - \frac{1}{4\pi}\oint_S \Phi_s\frac{\partial G_D}{\partial n'}dS'. \tag{3}$$

Since q is at $y' = b/2$ I expect to have two different expressions for the potential, one for $y \leq b/2$ and another for $y \geq b/2$.

Let us first calculate the volume integral which would give us the potential for the charge if all the surfaces were grounded. Because of Eq.(2) we obtain:

$$\Phi_V(x,y) = \frac{2q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\sin\frac{n\pi}{2}\frac{\sinh[(b-y_>)\frac{n\pi}{a}]\sinh(y_<\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})}, \tag{4}$$

where $y_>(y_<)$ is the larger (smaller) between y and $b/2$.

Now let's calculate the surface integral that would give the potential due to the surface potentials in the absence of charges.

The surface integral is given by:

$$-\frac{1}{4\pi}\oint_S \Phi_s\frac{\partial G_D}{\partial n'}dS'. \tag{5}$$

In this case the "surface" is the line parallel to the x-axis at $y = b$ for $0 \leq x \leq a$. The normal is $\hat{n}' = \hat{y}'$. Thus, we have to take the derivative with respect to y' and evaluate it at $y' = b$. Notice that at this surface always $y \leq y'$ then $y' = y_>$ and $y = y_<$.

Then,

$$\frac{\partial G_D}{\partial n'}|_S = \frac{\partial G_D}{\partial y'}|_{y'=b} = 8\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}(-\frac{n\pi}{a})\frac{\cosh[(b-y')\frac{n\pi}{a}]|_{y'=b}\sinh(y\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})} =$$

$$-\frac{8\pi}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\frac{\sinh(y\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})}. \tag{6}$$

Then,

$$\Phi_s(x,y) = \frac{2V}{a}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}\sinh(y\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})}\int_0^a dx'\sin\frac{n\pi x'}{a}. \tag{7}$$

The integral over x' gives $2a/n\pi$ for n odd and it vanishes for n even. Then

$$\Phi_s(x,y) = \frac{4V}{\pi}\sum_{j=0}^{\infty}\frac{\sin\frac{(2j+1)\pi x}{a}\sinh(\frac{(2j+1)\pi y}{a})}{(2j+1)\sinh(\frac{b(2j+1)\pi}{a})}. \tag{8}$$

Then combining both parts we found that

$$\Phi(x,y)=\frac{2q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\sin\frac{n\pi}{2}\frac{\sinh[(b-y_>)\frac{n\pi}{a}]\sinh(y_<\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})}+\frac{4V}{\pi}\sum_{j=0}^{\infty}\frac{\sin\frac{(2j+1)\pi x}{a}\sinh(\frac{(2j+1)\pi y}{a})}{(2j+1)\sinh(\frac{b(2j+1)\pi}{a})}. \tag{9}$$