Homework #12

Problem 7:

$$-D\nabla_r^2\phi(\mathbf{r}) + \kappa^2 D\phi(\mathbf{r}) = Q\delta(\mathbf{r}) \tag{1}$$

In k-space:

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \phi(\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}}d^3k, \tag{2}$$

and

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{k}.\mathbf{r}}d^3k, \tag{3}$$

Plugging (2) and (3) in (1):

$$-\frac{D}{(2\pi)^{3/2}}\int -\kappa^2\phi((\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}}d^3k + \frac{\kappa^2 D}{(2\pi)^{3/2}}\int \phi((\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}}d^3k = \frac{Q}{(2\pi)^3}\int e^{-i\mathbf{k}.\mathbf{r}}d^3k. \tag{4}$$

For each $\mathbf{k} = (k, \theta, \phi)$ we have:

$$D\kappa^2\phi(\mathbf{k}) + \kappa^2 D\phi(\mathbf{k}) = \frac{Q}{(2\pi)^3}. \tag{5}$$

Solving (5) we obtain:

$$\phi(\mathbf{k}) = \frac{Q}{(2\pi)^{3/2}D(\kappa^2 + k^2)} \tag{6}$$

Replacing (6) in (2) and integrating taking the z-axis parallel to $\mathbf{k}$ we obtain:

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_0^\infty \frac{k^2 Q}{k^2+\kappa^2}dk\int_{-1}^{1} d(cos\theta)e^{-ikrcos\theta}\int_0^{2\pi} d\phi =$$

$$\frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2 Q 2 sin kr}{(k^2+\kappa^2)kr}dk =$$

$$\phi(\mathbf{r}) = \frac{Q}{4\pi D}\frac{e^{-r\kappa}}{r}. \tag{8}$$