Homework #5

**Problem 4 - 4.1.11**:

We need to proove that $K_{ij}$ is a tensor knowing that $A^{jk}$ and ${B_i}^k$ are tensors.
In S:

$$K_{ij}A^{jk} = {B_i}^k. \tag{1}$$

In S':

$$K'_{ij}A'^{jk} = {B'_i}^k. \tag{2}$$

Since we know that ${B_i}^k$ is a tensor

$$ {B'_i}^k = \frac{\partial x^l}{\partial x'^i}\frac{\partial x'^k}{\partial x^m}{B_l}^m, \tag{3}$$

Using Eq.(1) to replace $B_l^m$ in Eq.(3) we obtain:

$$ {B'_i}^k = \frac{\partial x^l}{\partial x'^i}\frac{\partial x'^k}{\partial x^m}K_{lr}A^{rm}. \tag{4}$$

Since $A$ is a tensor we know that

$$A^{rm} = \frac{\partial x^r}{\partial x'^j}\frac{\partial x^m}{\partial x'^k}A'^{jk}. \tag{5}$$

Replacing Eq.(5) in Eq.(4) we obtain:

$$ {B'_i}^k = \frac{\partial x^l}{\partial x'^i}\frac{\partial x'^k}{\partial x^m}K_{lr}\frac{\partial x^r}{\partial x'^j}\frac{\partial x^m}{\partial x'^k}A'^{jk}. \tag{6}$$

Comparing Eq.(2) with Eq.(6) we get:

$$K'_{ij}A'^{jk} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x'^k}{\partial x^m}K_{lr}\frac{\partial x^r}{\partial x'^j}\frac{\partial x^m}{\partial x'^k}A'^{jk}. \tag{7}$$

Rearranging terms in Eq.(7) we get:

$$(K'_{ij} - \frac{\partial x^l}{\partial x'^i}\frac{\partial x'^k}{\partial x^m}\frac{\partial x^r}{\partial x'^j}\frac{\partial x^m}{\partial x'^k}K_{lr})A'^{jk} = 0. \qquad (7)$$

Since $A$ is a non-zero arbitrary tensor we know that its coefficient has to vanish to satisfy Eq.(7) then:

$$K'_{ij} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^r}{\partial x'^j}K_{lr}, \tag{8}$$

where we have used that $\frac{\partial x'^k}{\partial x^m}\frac{\partial x^m}{\partial x'^k} = 1$. Eq.(8) shows that $K$ transforms like a covariant tensor of rank 2, then, $K_{ij}$ is a tensor.