Homework #8

Problem 6:

The easiest way to solve this problem is by decomposing it into two problems and, using the principle of superposition, adding both solutions.

Thus, let's first solve the problem of a charge q located at $(0,d)$ in the region of space defined by $-\infty \leq x \leq \infty$ and $0 \leq y \leq a$ with the potential $\phi(x,y)$ set to zero on all the surfaces, i.e., $y=0$ and $y=a$, and at $x \to \pm\infty$.

In this case we need to divide the space in two regions because Laplace's equation is not valid at the location of q. Thus we define region I for $x \leq 0$ and region II for $x \geq 0$ and we propose:

$$\Phi^I(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{\pi n y}{a}) e^{\frac{\pi n x}{a}}, \tag{1}$$

and

$$\Phi^{II}(x,y) = \sum_{n=1}^{\infty} B_n \sin(\frac{\pi n y}{a}) e^{\frac{-\pi n x}{a}}, \tag{2}$$

where I have used that since $\Phi(x,0)=0$ we have to use $\sin(\alpha y)$ rather than $\cos(\alpha y)$ along the y direction and since $\Phi(x,a)=0$, $\alpha = \frac{\pi n y}{a}$. Since $\lim_{x\to-\infty}\Phi(x,y)=0$ only the positive real exponential has to appear in $\Phi^I(x,y)$ and since $\lim_{x\to\infty}\Phi(x,y)=0$ only the negative real exponential has to appear in $\Phi^{II}(x,y)$. Now we are going to use the boundary conditions at $x=0$ to determine the values of A_n and B_n.

We know that at $x=0$ the potential is continuous so

$$\Phi^I(0,y) = \Phi^{II}(0,y), \tag{3}$$

and since the normal component of the electric field $E_n = -\frac{\partial\Phi(x,y)}{\partial n} = -\frac{\partial\Phi(x,y)}{\partial x}$ since the normal is x in this case, has a jump across the surface equal to $\frac{\sigma}{\epsilon_0}$ and $\sigma = q\delta(y-d)$ is the surface density of charge at $x=0$ we obtain

$$-\frac{\partial\Phi^{II}(x,y)}{\partial x}|_{x=0} + \frac{\partial\Phi^{I}(x,y)}{\partial x}|_{x=0} = \frac{q\delta(y-d)}{\epsilon_0}. \tag{4}$$

From Eq.(3) we obtain that

$$A_n = B_n, \tag{5}$$

and from Eq.(4) we obtain that

$$\sum_{n=1}^{\infty} \frac{\pi n}{a} A_n \sin\frac{\pi n y}{a} = \frac{q\delta(y-d)}{\epsilon_0}. \tag{6}$$

Now let's multiply both sides of Eq.(6) by $\sin\frac{\pi m y}{a}$ and integrate over y in the interval $(0,a)$. On the left hand side the orthogonality of the sine gives $\delta_{nm}\frac{a}{2}$ and on the right hand side we obtain $\frac{q\sin\frac{\pi n d}{a}}{\epsilon_0}$ then

$$A_n = \frac{q}{n\pi\epsilon_0}\sin\frac{\pi n d}{a}. \tag{7}$$

Then we find that

$$\Phi^I(x,y) = \frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n d}{a}\sin\frac{\pi n y}{a}e^{\frac{\pi n x}{a}}, \tag{8}$$

$$\Phi^{II}(x,y) = \frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n d}{a}\sin\frac{\pi n y}{a}e^{\frac{-\pi n x}{a}}, \tag{9}$$

which can be combined as:

$$\Phi(x,y)=\frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi nd}{a}\sin\frac{\pi ny}{a}e^{\frac{-\pi n(x_>-x_<)}{a}}, \tag{10}$$

where $x_>$ $(x_<)$ is the larger (smaller) between x and 0.

Now we need to consider the second problem, i.e., find the potential in the charge free region defined by $-\infty \leq x \leq \infty$ and $0 \leq y \leq a$ with the potential $\phi(x,y)$ set to zero at $y=0$ and to V at $y=a$. We know that in this situation the planes given by y constant define equipotentials and thus the potential is going to be independent of x. In addition we know that the field in between the two planes is going to be a constant given by $\mathbf{E}=\frac{-V}{a}\mathbf{j}$ then

$$\Phi^V(x,y)=\frac{Vy}{a}. \tag{11}$$

Notice that in Eq.(11) the form of the solution to Laplace's equation is not of the form $e^{\pm\alpha y}$ or $e^{\pm i\alpha y}$ as we found in class. The reason for this is that by symmetry we see that the solution cannot depend on x this means that when we solve

$$\nabla^2\Phi=0 \tag{12}$$

we propose

$$\Phi(x,y)=X(x)Y(y)=Y(y), \tag{13}$$

i.e. $X(x)=1$ for all x. Then, replacing (13) in (12) we obtain:

$$\frac{\partial^2 Y}{\partial y^2}=0, \tag{14}$$

which has a solution of the form

$$Y(y)=Ay. \tag{15}$$

Using the boundary condition we find that $A=\frac{V}{a}$ and then we obtain the result displayed in Eq.(11).

Then the solution to the total problem is the sum of the two solutions, i.e.,Eq.(10)+Eq.(11):

$$\Phi(x,y)=\frac{q}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{e^{-\frac{\pi n}{a}(x_>-x_<)}}{n}\sin\frac{n\pi y}{a}\sin\frac{n\pi d}{a}+\frac{Vy}{a}. \tag{16}$$