Homework #8

Problem 4 - 9.4.2:

We need to consider the equation

$$\nabla^2\psi + (k^2 + f(\rho) + \frac{1}{\rho^2}g(\phi) + h(z))\psi, \tag{1}$$

and show that it can be solved by separation of variables in cylindrical coordinates. Thus, we propose:

$$\psi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z), \tag{2}$$

and we replace (2) in (1) and divide by (2):

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho}) + f(\rho) + \frac{1}{Q\rho^2}\frac{\partial^2 Q}{\partial\phi^2} + \frac{1}{\rho^2}g(\phi) + k^2 + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + h(z) = 0. \tag{3}$$

We see that the last two terms only depend on z so we can equate them to a constant that we call $-\alpha^2$ which means that

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho}) + f(\rho) + \frac{1}{Q\rho^2}\frac{\partial^2 Q}{\partial\phi^2} + \frac{1}{\rho^2}g(\phi) + k^2 = \alpha^2, \tag{4}$$

to satisfy (3). Multiplying (4) by ρ^2 we obtain:

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho}) + \rho^2 f(\rho) + (k^2 - \alpha^2)\rho^2 = -\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} - g(\phi) = \beta^2. \tag{5}$$

In (5) we see that the equation is again separated and we had equated each term to the separation constant β^2.