For all this to make sense, once we normalize to 1 at t=0 the normalization must remain. Otherwise particles will be created or vanished varying time. Is this true?

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2\,dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

But $\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi$ and $\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$

Obtained by multiplying all terms in Sch. Eq. by $-i/\hbar$

The terms with V cancel out after addition.

Check!

Check!

Check!

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]$$

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \int_{-\infty}^{+\infty}\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]dx$$

$$= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty} = 0$$

If $\psi \rightarrow 0$ as $x \rightarrow (\pm)$ infinity

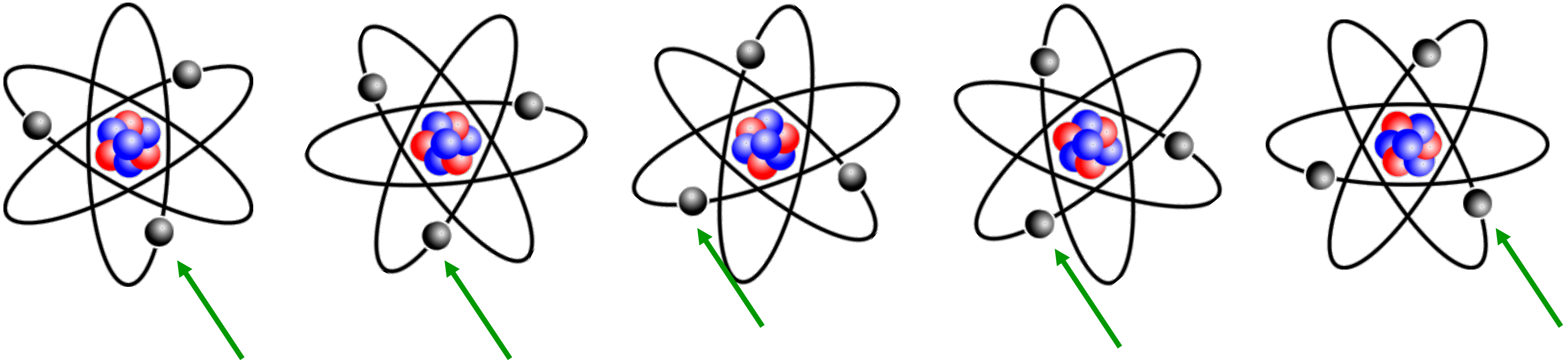
If ψ is normalized at t=0, it remains normalized at all times.
Crucial for all this to make sense!

Expectation value of x

$$\langle x\rangle = \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2\,dx$$

Note: <x> can be time dependent.

Interpretation: <x> is the average of measurements performed on an ensemble of identical systems.



Expectation value of momentum p

See two pages back

$$\langle v \rangle = \frac{d\langle x\rangle}{dt} = \int x \frac{\partial}{\partial t}|\Psi|^2\, dx = \frac{i\hbar}{2m}\int \underbrace{x}_{f} \frac{\partial}{\partial x}\underbrace{\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)}_{g} dx$$

(See Chs 2 and 3)

Integration by parts (back cover of book):

$$\int_a^b f \frac{dg}{dx}\, dx = -\int_a^b \frac{df}{dx} g\, dx + fg\Big|_a^b$$

df/dx=1

g

Check!

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{2m}\int \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx$$

By parts again

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int \Psi^* \frac{\partial \Psi}{\partial x}\, dx$$

Check!

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int \left(\Psi^* \frac{\partial \Psi}{\partial x}\right) dx$$

In summary, for <*x*> and <*p*> we find

$$\langle x \rangle = \int \Psi^* \underbrace{(x)} \Psi \, dx \qquad \langle p \rangle = \int \Psi^* \underbrace{\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)} \Psi \, dx$$

x "operator" is just "multiply by x

p "operator" is more complicated!

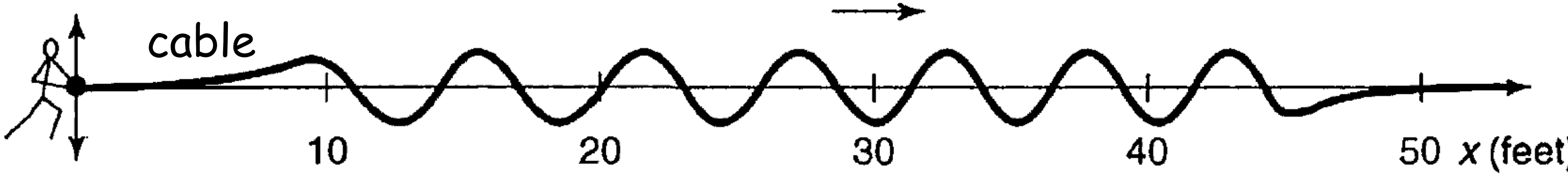
Many other operators are functions of x and p.
For instance, for the kinetic energy $T = p^2/2m$ use:

$$p^2 = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

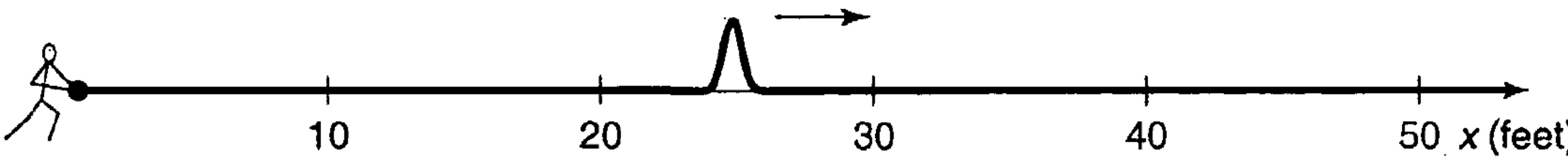
By this procedure a "**dictionary**" between classical and quantum quantities can be established.

Preliminaries to the uncertainty principle

Caution: this is not an independent principle but it arises entirely from Sch. Eq. (see Ch. 3). Thus, if you do the calculations right, it is always satisfied. But intuitively it is interesting to discuss it.



Wavelength? Clear; Position? Unclear



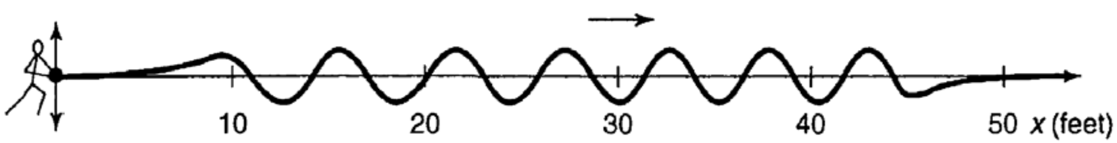
Wavelength? Unclear; Position? Clear

This is true for any wave-like phenomenon, thus it has to apply to the Sch. Eq. somehow as well.

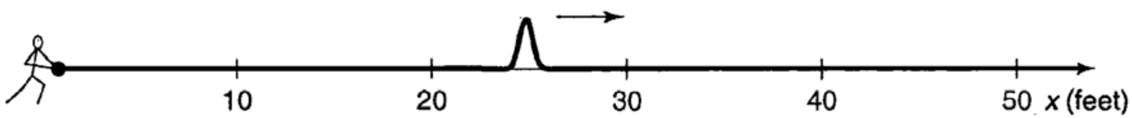
De Broglie formula (2 years before Sch. Eq.) said that electrons have wave-like features, like photons do:

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

Thus, if wavelength is known accurately, p is known accurately. If wavelength is unknown, p is unknown.



Momentum? Sharp
Position? Not sharp



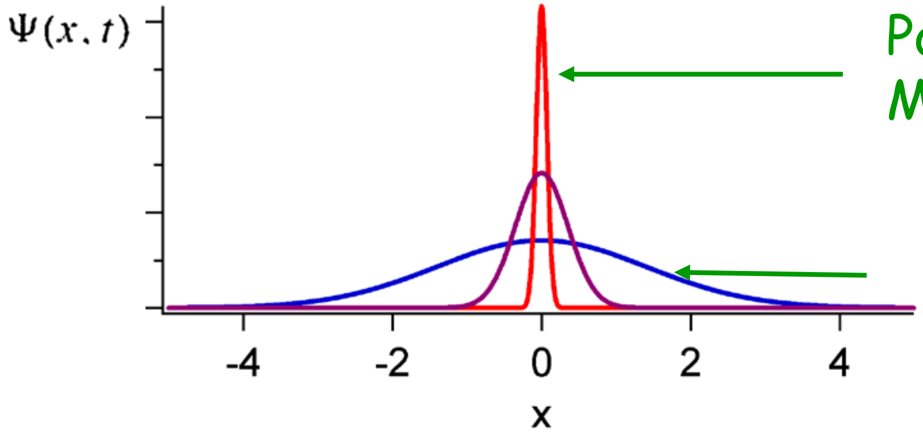
Momentum? Not sharp
(if you Fourier decompose a spike, it has all k values!)
Position? Sharp

We will prove later (not a new law, but it is consequence of Sch. Eq.) that the standard deviations satisfy:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$



Position? Sharp σ_x=0 if δ function
Momentum? Totally unknown $\sigma_p=\infty$

Position? Less sharp σ_x>0
Momentum? Less unknown $\sigma_p<\infty$