The same holds for all components:

$$[L^2, L_x] = 0, \quad [L^2, L_y] = 0, \quad [L^2, L_z] = 0$$

Because L^2 commutes with at least one component (usually chosen to be L_z) then we should find **eigenstates of both operators simultaneously**.
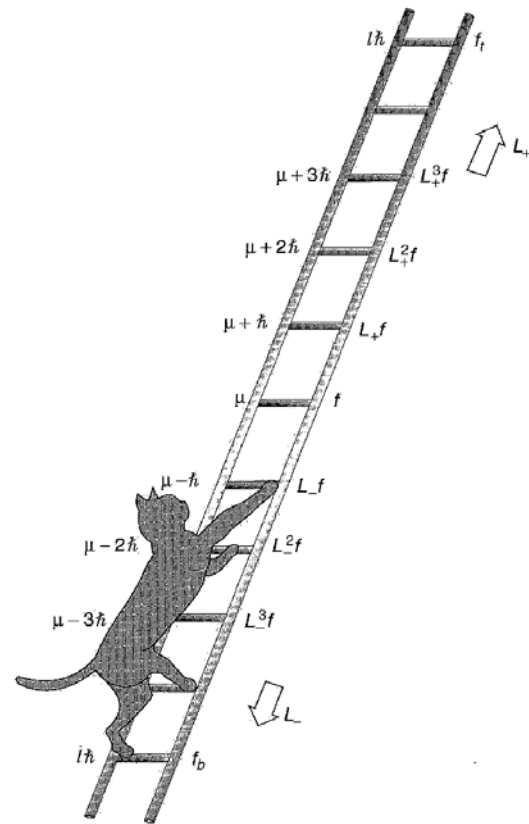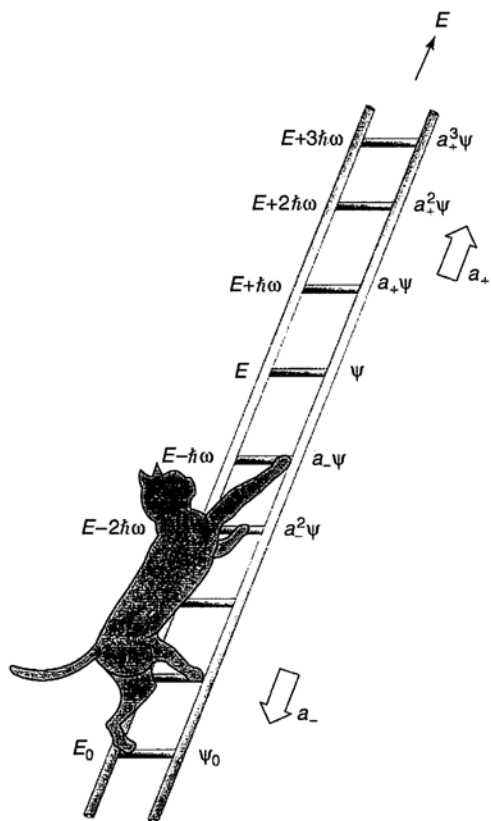
$$L^2 f = \lambda f \qquad L_z f = \mu f$$

Then our mission is to find λ and μ and f.

To solve this problem we will use a procedure very similar to that of the Harmonic Oscillator with the *lowering and raising operators.*

In Ch. 2 we defined the raising and lowering operators.

$$a_{\pm} \equiv \frac{1}{\sqrt{2\hbar m\omega}} (\mp ip + m\omega x)$$
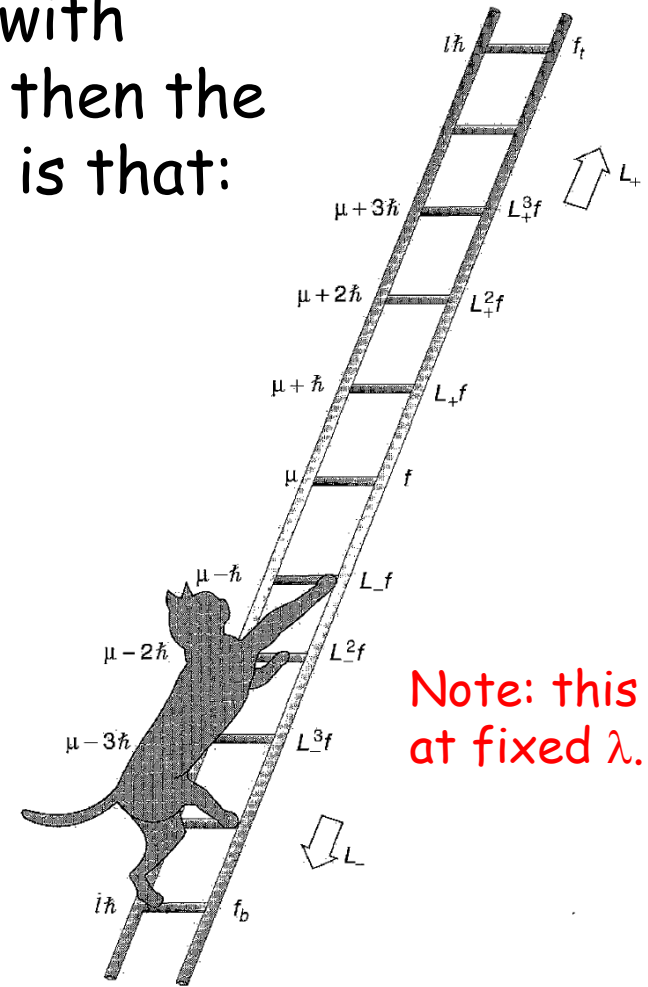
$$L_{\pm} \equiv L_x \pm iL_y$$

If *f* is eigenfunction of L^2 and L_z with eigenvalues λ and μ, respectively, then the claim (proof on page 158, similar as in Ch2) is that:

$$L^2(L_\pm f) = \lambda(L_\pm f)$$

$$L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$$

L_+ is the raising operator and L_- the lowering operator.

Note: this is at fixed λ.

But like in Ch.2 this cannot go on forever. Eventually the projection, positive or negative, will be larger than the vector itself.

3

At the top value, let us call the L_z max eigenvalue $\hbar l$

$$L_z f_t = \hbar l f_t; \qquad L^2 f_t = \lambda f_t$$

Useful identity:

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x)$$

$$= L^2 - L_z^2 \mp i(i\hbar L_z) \quad \text{Or, just reorganizing:}$$

$$\boxed{L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$$

Two identities: use one.

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t =$$

$$= (0 + \hbar^2 l^2 + \hbar^2 l) f_t = \hbar^2 l(l+1) f_t$$

$$\boxed{\lambda = \hbar^2 l(l+1)}$$

On page 160, you can find the results starting at the bottom rung.

4

Too tedious to continue with all the details but you have the essence of the reasoning already.
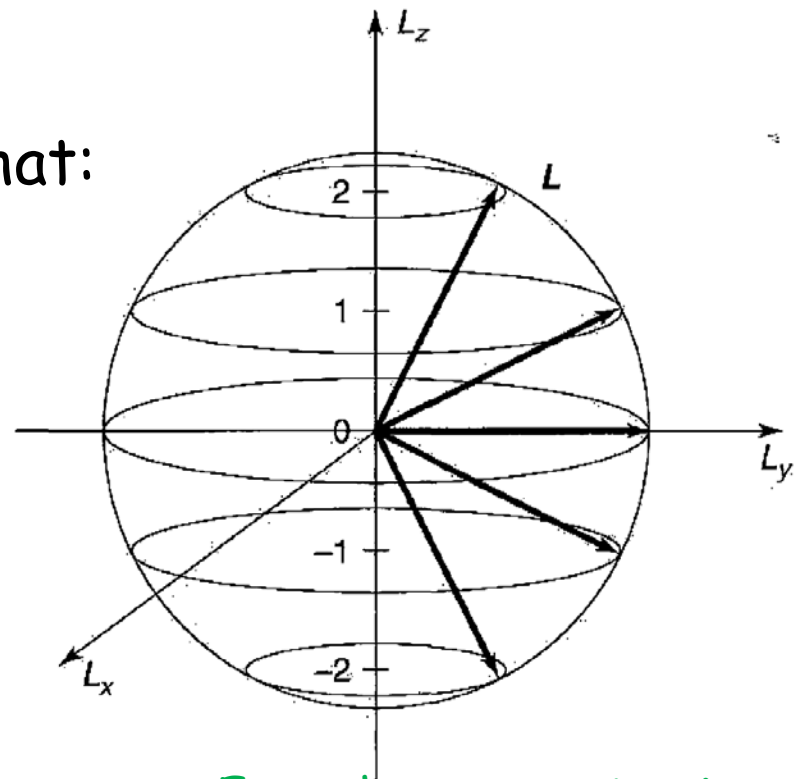
$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m$$

Analyzing now the bottom of the chain of states it can be shown that:

$$m = -l, \ -l+1, \ \ldots, \ l-1, \ l$$

thus $l = 0, \ 1/2, \ 1, \ 3/2, \ \ldots$

Note that *l* can be integer or half-integer mathematically speaking. For instance, l=3/2, can have m=3/2,1/2,-1/2,-3/2. All math satisfied! → Hint: spin!

Even the max projection has Lx and Ly uncertainty (read book)

5

In the previous page we found the meaning of "*l*" and "*m*" in the quantum numbers *(n,l,m).*
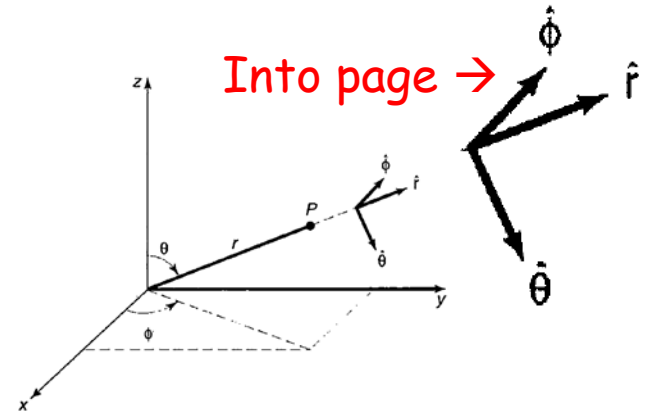
The eigenvalues of the **L**^{2} operator were ℏ^{2}l(l+1) and those of the **L**$_z$ operator were ℏm.

Now we need the eigenfunctions ... (more difficult)

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \mathbf{p} \rightarrow \frac{\hbar}{i} \nabla \qquad \mathbf{L} = (\hbar/i)(\mathbf{r} \times \nabla)$$
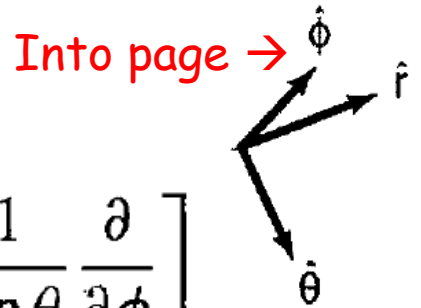
classical   classical to QM   QM angular momentum

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Into page →

(not obvious) gradient operator in spherical coordinates. Search in some math book.

6

Remember that $\mathbf{r} = r\hat{r}$. Then:

$$\mathbf{L} = \frac{\hbar}{i}\left[r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + (\hat{r}\times\hat{\theta})\frac{\partial}{\partial\theta} + (\hat{r}\times\hat{\phi})\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$= 0 \qquad\qquad = \hat{\phi} \qquad\qquad = -\hat{\theta}$$

We arrive to $\mathbf{L} = \dfrac{\hbar}{i}\left(\hat{\phi}\dfrac{\partial}{\partial\theta} - \hat{\theta}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\phi}\right)$

We can now rewrite in terms of the unit vectors in Cartesian coordinates using:

$$\hat{\theta} = (\cos\theta\cos\phi)\hat{\imath} + (\cos\theta\sin\phi)\hat{\jmath} - (\sin\theta)\hat{k}$$

$$\hat{\phi} = -(\sin\phi)\hat{\imath} + (\cos\phi)\hat{\jmath},$$

By mere replacement (easy) we arrive to:

$$L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_y = \frac{\hbar}{i}\left(+\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

Why do we care so much about *x,y,z* components instead of θ,ϕ,r components? Because we have all commutators etc etc written in terms of x,y,z components from previous pages …

The very important raising and lowering operators then become (again, easy):

$$L_\pm = L_x \pm iL_y = \pm \hbar e^{\pm i\phi} \left(\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi} \right)$$

Using a relation (easy) derived some pages back:

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

we can deduce an expression (not as easy) for L^2 in spherical coordinates:

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

So we finally arrived to the differential equation we wish to solve to find the eigenfunctions:

$$L^2 f_l^m = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] f_l^m = \hbar^2 l(l+1) f_l^m$$

L^2 from previous page   from some pages back

HOWEVER, this equation happens to be the SAME "angular equation" that we derived at the start of Chapter 4 when we were trying "separation of variables" to solve the Sch. Eq. (see Eq.[4.17] book):

10

The "angular equation" was (a mere division by -Y ℏ² left and right is the only difference):

$$\frac{1}{Y} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} \right\} = -l(l+1)$$

We showed early in the chapter that the solutions were the Spherical Harmonics, so there is no further work to do!

Same for the other (much easier) equation (left as exercise):

$$L_z f_l^m = \frac{\hbar}{i} \frac{\partial}{\partial\phi} f_l^m = \hbar m f_l^m$$

In summary: the Spherical Harmonics, that we studied in detail before, are the eigenfunctions of the L^2 and L_z operators.

The eigenfunctions of the Hamiltonian of the Hydrogen atom

$$H\psi = E\psi, \quad L^2\psi = \hbar^2 l(l+1)\psi, \quad L_z\psi = \hbar m\psi$$

were already eigenfunctions of L^2 and L_z

Warning: we know spherical harmonics works well for integer l. For half-integer l, the story will be very different ...

This completes the logic: the "*l*" and "*m*" quantum numbers introduced mathematically during the separation of variables procedure have a profound physical meaning.

The eigenvalues of L^2 are $\hbar^2 l(l+1)$ and those of L_z are $\hbar m$.

Note that there is a $l(l+1)$ not a l^2. At large "*l*" the difference is small but at say $l=1$ it is not small.