How about the ground state wave function?
Remember:

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\,Y_l^m(\theta,\phi)$$

$$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho) \qquad u(\rho) = \rho^{l+1}e^{-\rho}v(\rho) \qquad \rho \equiv \kappa r$$

R(r)=u(r)/r

n=1 is the lowest energy (the most negative).
Since $n = j_{max} + l + 1$, then $l=0$, $j_{max} = 0$. Then

$$\psi_{100}(r,\theta,\phi) = R_{10}(r)Y_0^0(\theta,\phi)$$

with $R_{10}(r) = \frac{c_0}{a}e^{-r/a}$ because
the polynomial has only a
constant c_0.

$$Y_0^0(\theta,\phi) = 1/\sqrt{4\pi}$$

We then normalize, which means we fix the value of c_0 :

$$\int_0^\infty |R_{10}|^2 r^2\,dr = \frac{|c_0|^2}{a^2}\int_0^\infty e^{-2r/a}\,r^2\,dr = |c_0|^2\frac{a}{4} = 1$$

The very final result (celestial music ...) is:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

which, amazingly, is remarkably simple.

Consider now the excited states, starting with n=2:

$$E_2 = \frac{-13.6 \text{ eV}}{4} = -3.4 \text{ eV}$$

Because $n = j_{max} + l + 1$, then n=2 allows for $l=0$, $j_{max}= 1$ or $l=1$ ($m=-1,0,+1$), $j_{max}= 0$. This will be the 2s and three 2p's orbitals. Degeneracy 4 (in general, degeneracy is n^2).

(1) If $l=0$, $j_{max}= 1$, then $c_{jmax+1}=c_2=0$.
Use $c_1 = [2(j + l + 1 - n) / (j+1)(j+2l+2)]\, c_0$ with $n=2$, $j=0$ (because c_0 means $j=0$) and $l=0$, and you get $c_1 = - c_0$ so the polynomial becomes $v(\rho)=c_0(1-\rho)$ with c_0 again used to normalize.

$$Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$$

$$R_{20}(r) = \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a}$$

The factors 2 arise from $\rho = (1/an)r$ with n=2.

(2) If $l=1$, $j_{max}=0$, you get $c_{jmax+1}=c_1=0$ so the polynomial is only c_0 as it happens in the ground state.

Use the general formula $R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho)$ with $l=1$ and $n=2$.

$$R_{21}(r) = \frac{c_0}{4a^2} r e^{-r/2a}$$

$\rho = (1/an)r$ with $n=2$

The front factor comes from $\rho^2/r = (r/2a)^2/r$.

The spherical harmonics are:

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$$

$$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

In general, the polynomials $\nu(\rho)$ are called **associated Laguerre polynomials**

$$v(\rho) = L^{2l+1}_{n-l-1}(2\rho) \qquad \rho = (1/an)\, r$$

and there are tables with these polynomials.

Putting all together, for arbitrary *(n,l,m)* the normalized wave functions are:

$$\psi_{nlm} = \underbrace{\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l \left[L^{2l+1}_{n-l-1}(2r/na)\right]}_{R_{nl}(r)} Y^m_l(\theta, \phi)$$

$$R_{10} = 2a^{-3/2} \exp(-r/a)$$

$$R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2}\frac{r}{a}\right) \exp(-r/2a)$$

$$R_{21} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp(-r/2a)$$

$$R_{30} = \frac{2}{\sqrt{27}} a^{-3/2} \left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right) \exp(-r/3a)$$

$$R_{31} = \frac{8}{27\sqrt{6}} a^{-3/2} \left(1 - \frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right) \exp(-r/3a)$$

$$R_{32} = \frac{4}{81\sqrt{30}} a^{-3/2} \left(\frac{r}{a}\right)^2 \exp(-r/3a)$$

$$R_{40} = \frac{1}{4} a^{-3/2} \left(1 - \frac{3}{4}\frac{r}{a} + \frac{1}{8}\left(\frac{r}{a}\right)^2 - \frac{1}{192}\left(\frac{r}{a}\right)^3\right) \exp(-r/4a)$$

3 nodes

$$R_{41} = \frac{\sqrt{5}}{16\sqrt{3}} a^{-3/2} \left(1 - \frac{1}{4}\frac{r}{a} + \frac{1}{80}\left(\frac{r}{a}\right)^2\right) \frac{r}{a} \exp(-r/4a)$$

2 nodes

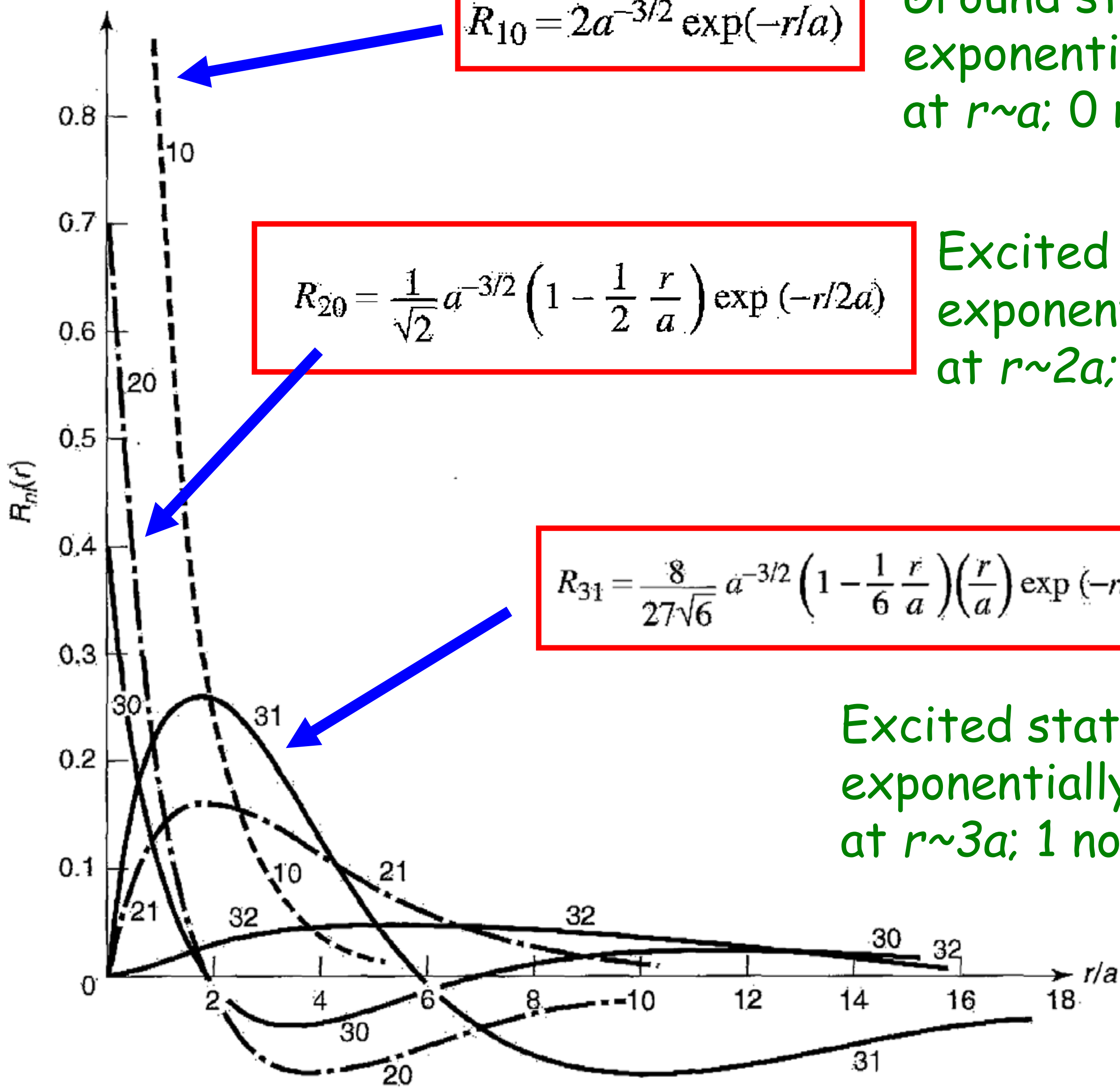
$$R_{42} = \frac{1}{64\sqrt{5}} a^{-3/2} \left(1 - \frac{1}{12}\frac{r}{a}\right)\left(\frac{r}{a}\right)^2 \exp(-r/4a)$$

1 nodes

$$R_{43} = \frac{1}{768\sqrt{35}} a^{-3/2} \left(\frac{r}{a}\right)^3 \exp(-r/4a)$$

0 nodes

Only l=0 are nonzero at the center r=0, as in the spherical well.



$$R_{10} = 2a^{-3/2}\exp(-r/a)$$

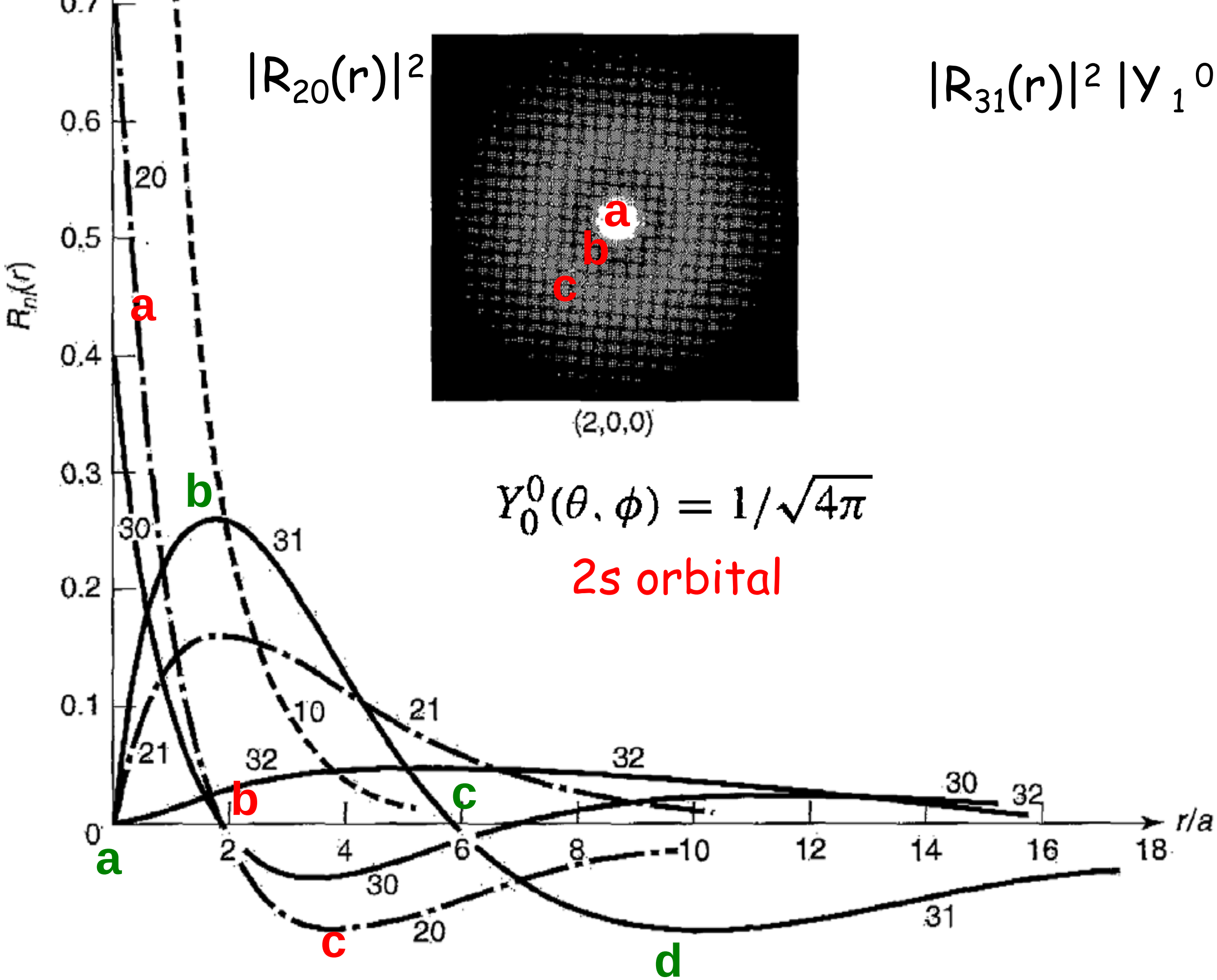
Ground state *n=1 l=0;* exponentially suppressed at *r~a;* 0 node

$$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\,\frac{r}{a}\right)\exp\,(-r/2a)$$

Excited state *n=2 l=0;* exponentially suppressed at *r~2a; 1 node.*

$$R_{31} = \frac{8}{27\sqrt{6}}\,a^{-3/2}\left(1 - \frac{1}{6}\,\frac{r}{a}\right)\left(\frac{r}{a}\right)\exp\,(-r/3a)$$

Excited state *n=3 l=1;* exponentially suppressed at *r~3a;* 1 node.

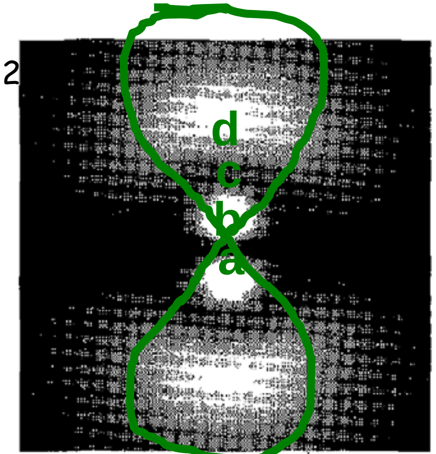


$|R_{20}(r)|^2$

$$Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$$

2s orbital

$|R_{31}(r)|^2 \; |Y_1^{\,0}|^2$



(3,1,0)

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$3p_z$ orbital