How about the ground state wave function?
Remember:

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

$$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho) \qquad u(\rho) = \rho^{l+1}e^{-\rho}v(\rho) \qquad \rho \equiv \kappa r$$

$R(r)=u(r)/r$

We start with $n=1$, the lowest energy (i.e. the most negative). Since $n = j_{max} + l + 1$, then $l=0$, $j_{max}= 0$. Then

$$\psi_{100}(r,\theta,\phi) = R_{10}(r)Y_0^0(\theta,\phi)$$

with $R_{10}(r) = \frac{c_0}{a}e^{-r/a}$ because the polynomial has only a constant c_0.

$$Y_0^0(\theta,\phi) = 1/\sqrt{4\pi}$$

We then normalize, which means we fix the value of c_0 :

$$\int_0^\infty |R_{10}|^2 r^2\, dr = \frac{|c_0|^2}{a^2} \int_0^\infty e^{-2r/a}\, r^2\, dr = |c_0|^2 \frac{a}{4} = 1$$

The very final result for the ground state of an electron in a H atom is:

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

which, amazingly, is remarkably simple.

Consider now the excited states, starting with $n=2$:

$$E_2 = \frac{-13.6 \text{ eV}}{4} = -3.4 \text{ eV}$$

Because $n = j_{max} + l + 1$, then $n=2$ allows for $l=0$, $j_{max}= 1$ or $l=1$ ($m=1,0,-1$), $j_{max}= 0$. Thus, degeneracy is 4 (in general, degeneracy is n^2).
This will be the *2s* and three *2p*'s orbitals.

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(2) If $l=0$, $j_{max}= 1$, then $c_{jmax+1}=c_2=0$.
Use $c_1 = [2(j + l + 1 - n) / (j+1)(j+2l+2)]\, c_0$ with $n=2$, $j=0$ (because c_0 means $j=0$) and $l=0$, and you get $c_1 = - c_0$ so the polynomial becomes $v(\rho)=c_0(1-\rho)$ with c_0 again used to normalize.

$$R_{20}(r) = \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a}$$

$$Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$$

The factors 2 arise from $\rho = (1/an)r$ with $n=2$.

(3) If $l=1$, $j_{max}= 0$, you get $c_{jmax+1}=c_1 = 0$ so the polynomial is only c_0, as for the ground state.

Use the general formula $R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho)$ with $l=1$ and $n=2$.

$$R_{21}(r) = \frac{c_0}{4a^2} r e^{-r/2a}$$

$\rho = (1/an)r$ with $n=2$

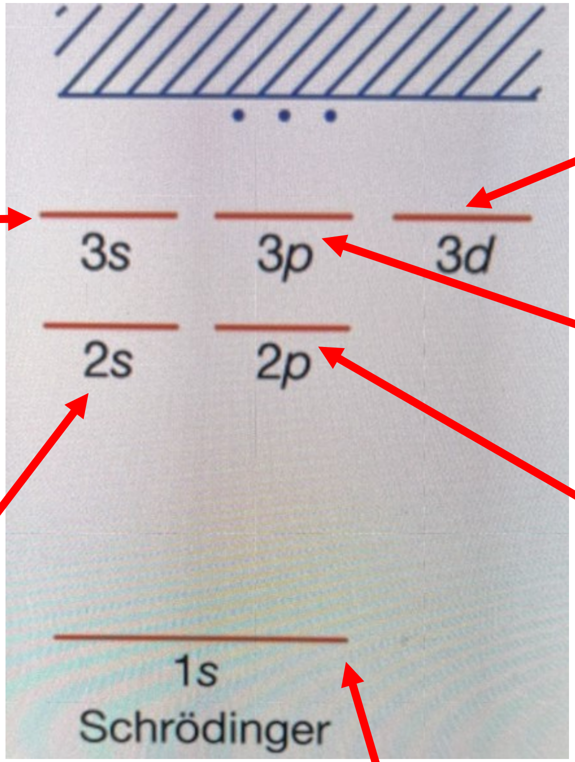
The front factor comes from $\rho^2/r = (r/2a)^2/r$.

The spherical harmonics are:

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$$

Visually: the familiar levels of the H-atom arise from playing with l and j_{max}



l=0 l=1 l=2

l=0, j_{max} =2, deg 1

l=0, j_{max} =1, deg 1

l=2, j_{max} =0, deg 5

l=1, j_{max} =1, deg 3

l=1, j_{max} =0, deg 3

l=0, j_{max} =0 (ground state), deg 1

In general, the polynomials ν(ρ) are called **associated Laguerre polynomials**

$$v(\rho) = L_{n-l-1}^{2l+1}(2\rho) \qquad \rho = (1/an)\, r$$

and there are tables with these polynomials.

Putting all together, for arbitrary *(n,l,m)* the normalized wave functions are:

$$\psi_{nlm} = \underbrace{\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}(2r/na)\right]}_{R_{nl}(r)} Y_l^m(\theta,\phi)$$

$$R_{10} = 2a^{-3/2}\exp(-r/a)$$

$$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)\exp(-r/2a)$$

$$R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}\exp(-r/2a)$$

$$R_{30} = \frac{2}{\sqrt{27}}a^{-3/2}\left(1-\frac{2}{3}\frac{r}{a}+\frac{2}{27}\left(\frac{r}{a}\right)^2\right)\exp(-r/3a)$$

$$R_{31} = \frac{8}{27\sqrt{6}}a^{-3/2}\left(1-\frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right)\exp(-r/3a)$$

$$R_{32} = \frac{4}{81\sqrt{30}}a^{-3/2}\left(\frac{r}{a}\right)^2\exp(-r/3a)$$

$$R_{40} = \frac{1}{4}a^{-3/2}\left(1-\frac{3}{4}\frac{r}{a}+\frac{1}{8}\left(\frac{r}{a}\right)^2-\frac{1}{192}\left(\frac{r}{a}\right)^3\right)\exp(-r/4a)$$

$$R_{41} = \frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(1-\frac{1}{4}\frac{r}{a}+\frac{1}{80}\left(\frac{r}{a}\right)^2\right)\frac{r}{a}\exp(-r/4a)$$

$$R_{42} = \frac{1}{64\sqrt{5}}a^{-3/2}\left(1-\frac{1}{12}\frac{r}{a}\right)\left(\frac{r}{a}\right)^2\exp(-r/4a)$$

$$R_{43} = \frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3\exp(-r/4a)$$

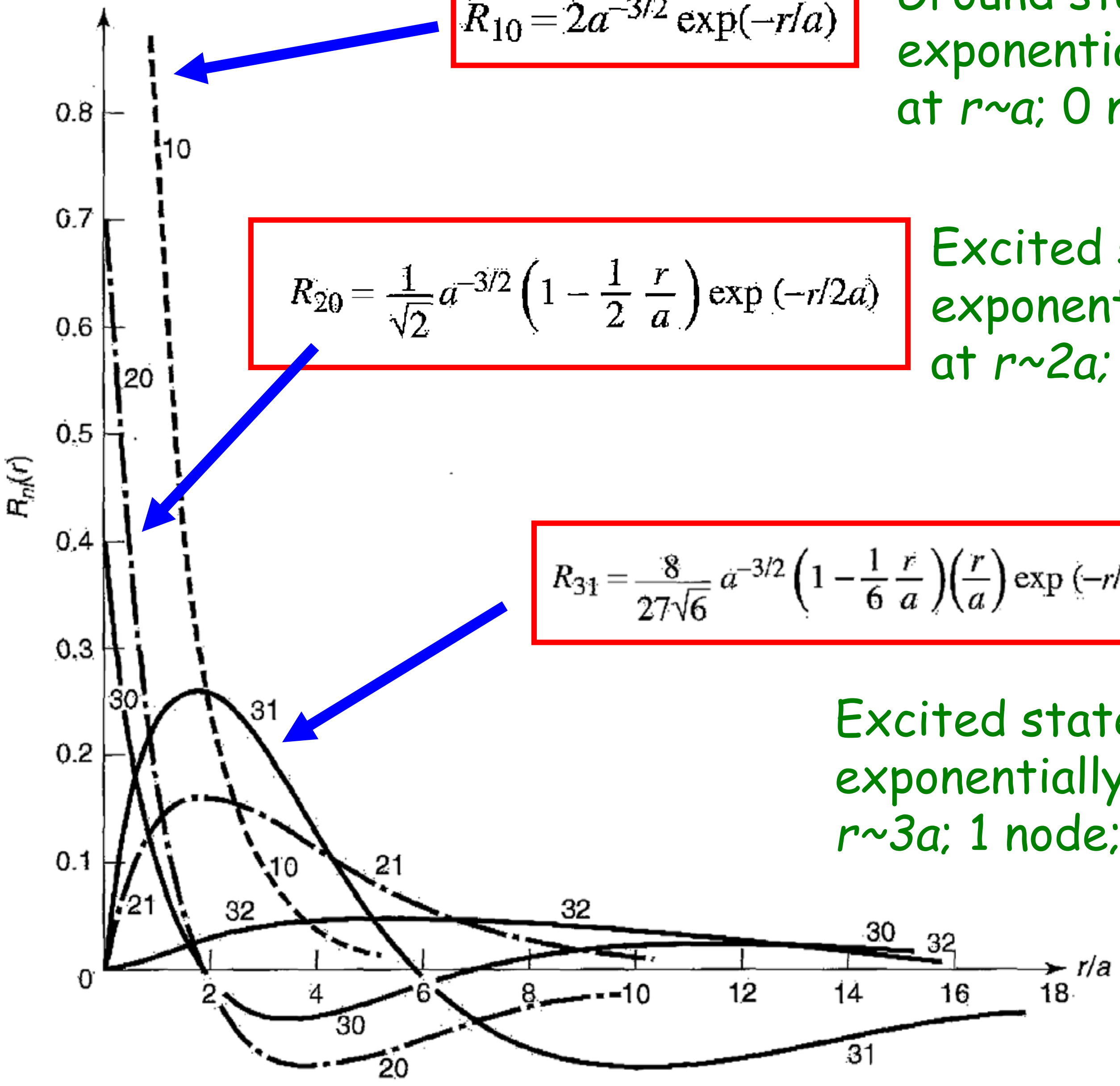
Only l=0 are nonzero at the center r=0, as it happened in the spherical well.

3 nodes

2 nodes

1 nodes

0 nodes



$$R_{10} = 2a^{-3/2}\exp(-r/a)$$

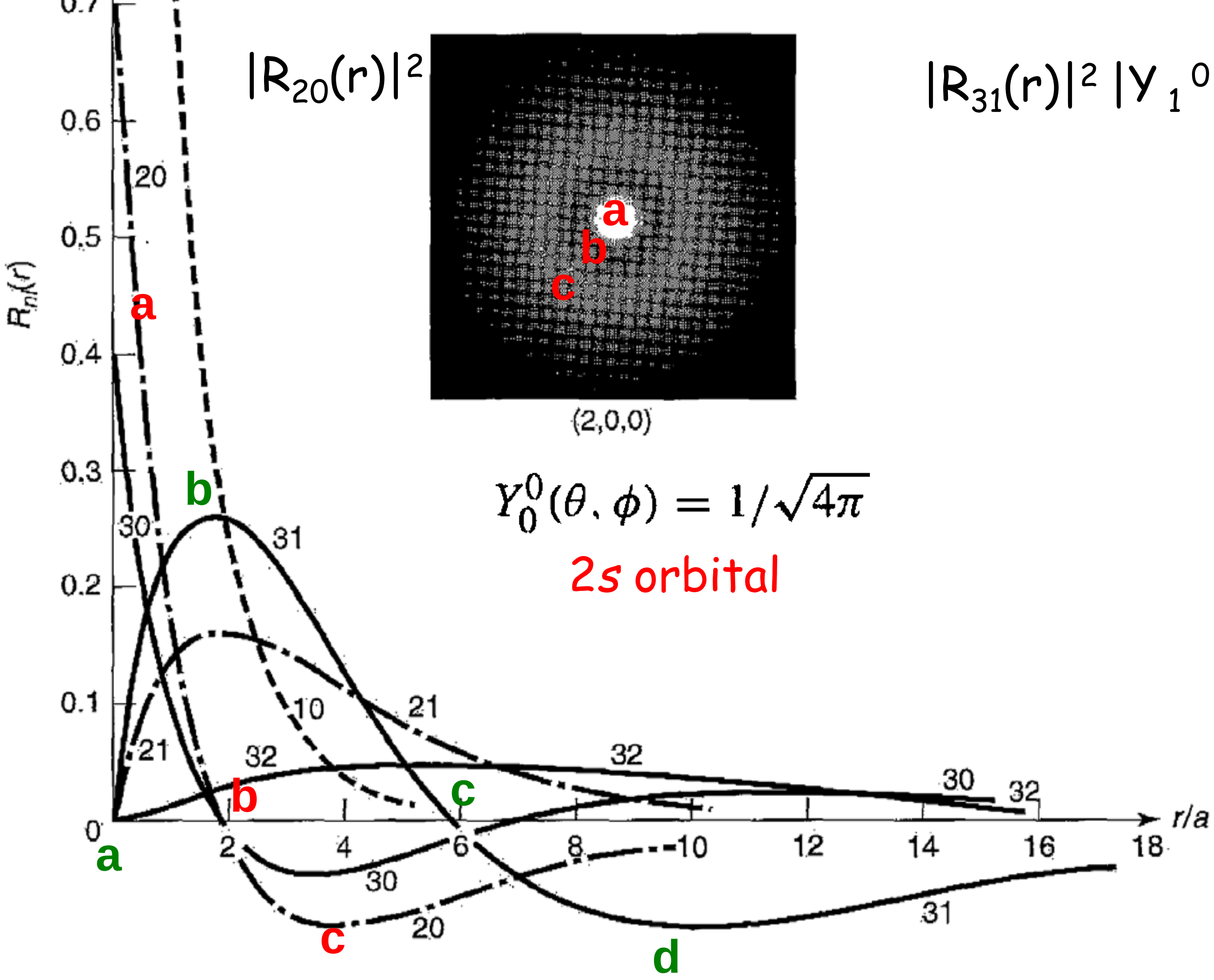
Ground state *n=1 l=0*; exponentially suppressed at *r~a*; 0 node

$$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)\exp(-r/2a)$$

Excited state *n=2 l=0*; exponentially suppressed at *r~2a; 1 node.*

$$R_{31} = \frac{8}{27\sqrt{6}}a^{-3/2}\left(1-\frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right)\exp(-r/3a)$$

Excited state *n=3 l=1*; exponentially suppressed at *r~3a*; 1 node; cancels at r=0.



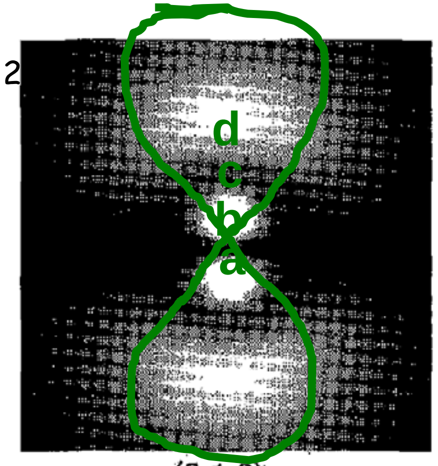
$|R_{20}(r)|^2$

(2,0,0)

$$Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$$

2s orbital

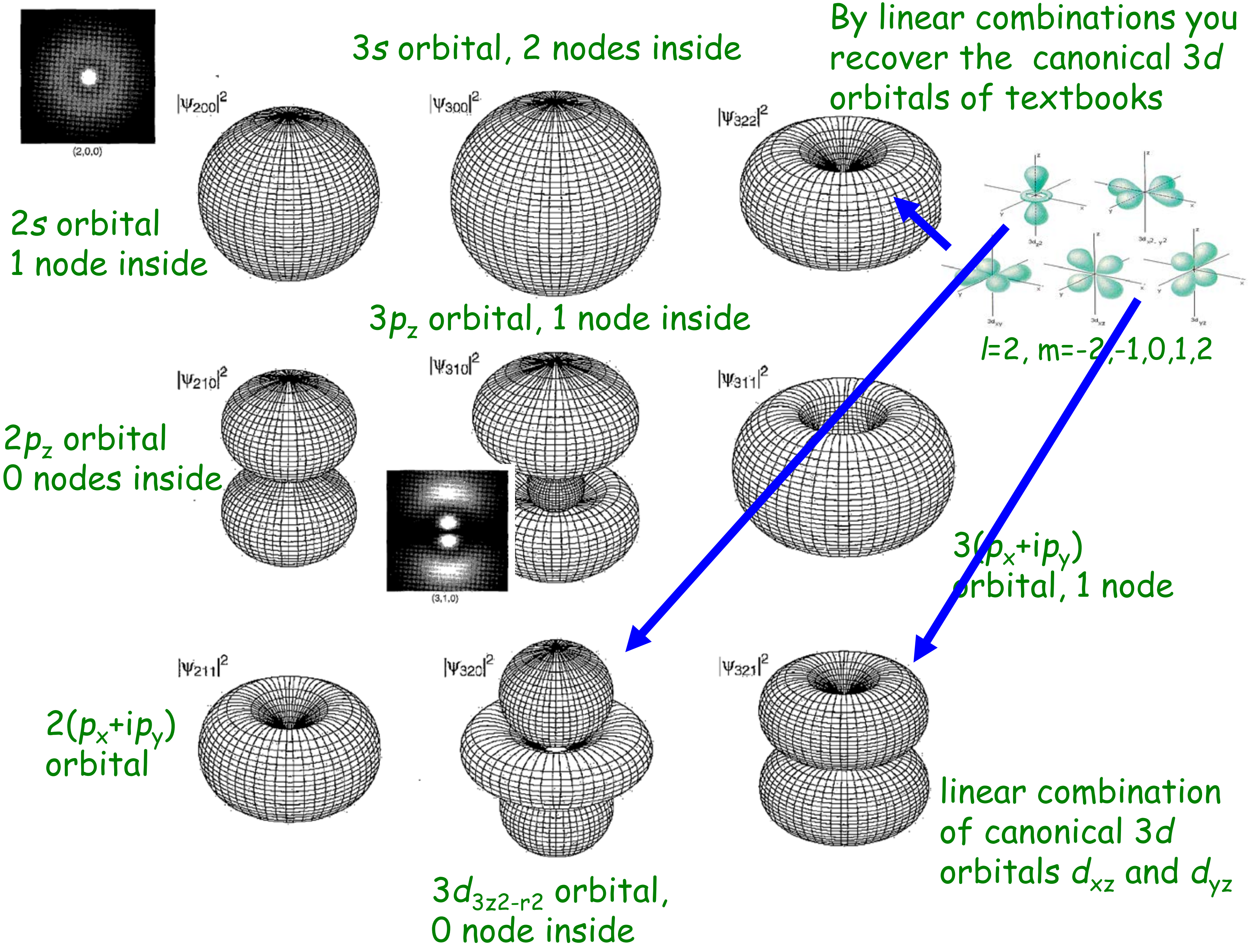
$|R_{31}(r)|^2 \, |Y_1^0|^2$



(3,1,0)

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$3p_z$ orbital


By linear combinations you recover the canonical 3d orbitals of textbooks
3s orbital, 2 nodes inside
2s orbital
1 node inside
(2,0,0)
|ψ200|2
|ψ300|2
|ψ322|2
3pz orbital, 1 node inside
l=2, m=-2,-1,0,1,2
|ψ210|2
|ψ310|2
|ψ311|2
2pz orbital
0 nodes inside
(3,1,0)
3(px+ipy)
orbital, 1 node
|ψ211|2
|ψ320|2
|ψ321|2
2(px+ipy)
orbital
linear combination of canonical 3d orbitals dxz and dyz
3d3z2-r2 orbital,
0 node inside

And as usual, the wave functions are orthonormal:

$$\int \psi^*_{nlm}\,\psi_{n'l'm'}\,r^2 \sin\theta\,dr\,d\theta\,d\phi = \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

Because of radial equation.

Because of spherical harmonics.