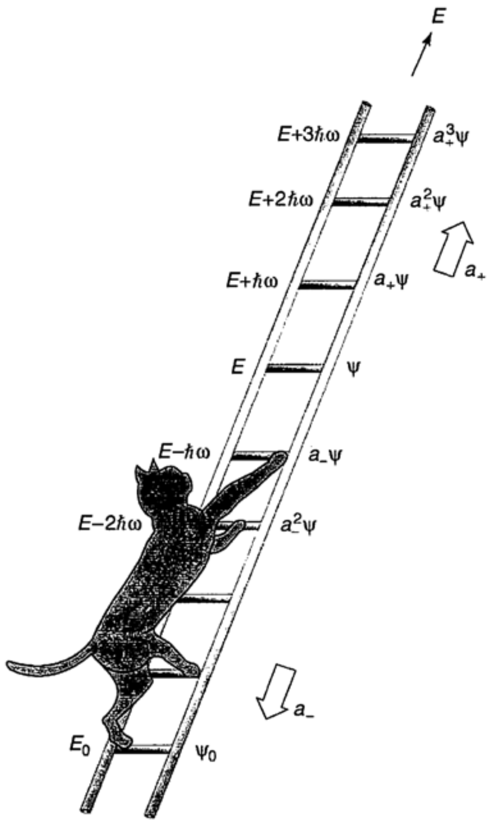
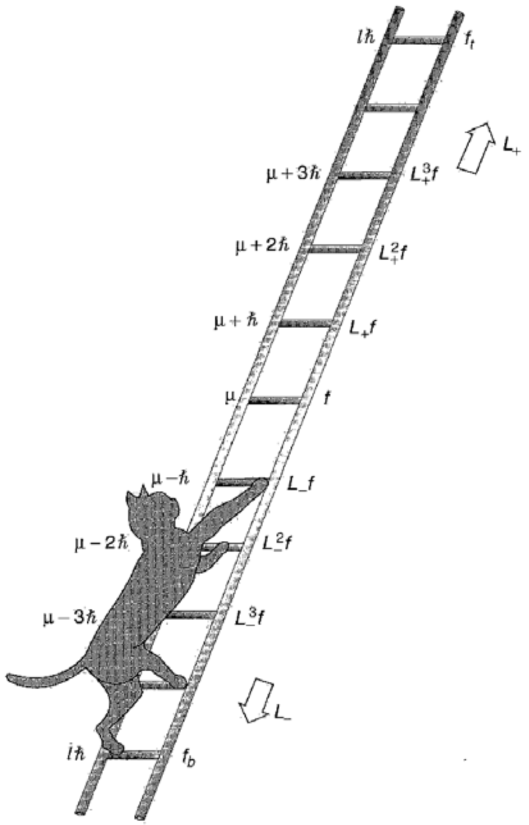
In Ch. 2 we defined the raising and lowering operators.

$$a_{\pm} \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

$$L_{\pm} \equiv L_x \pm iL_y$$

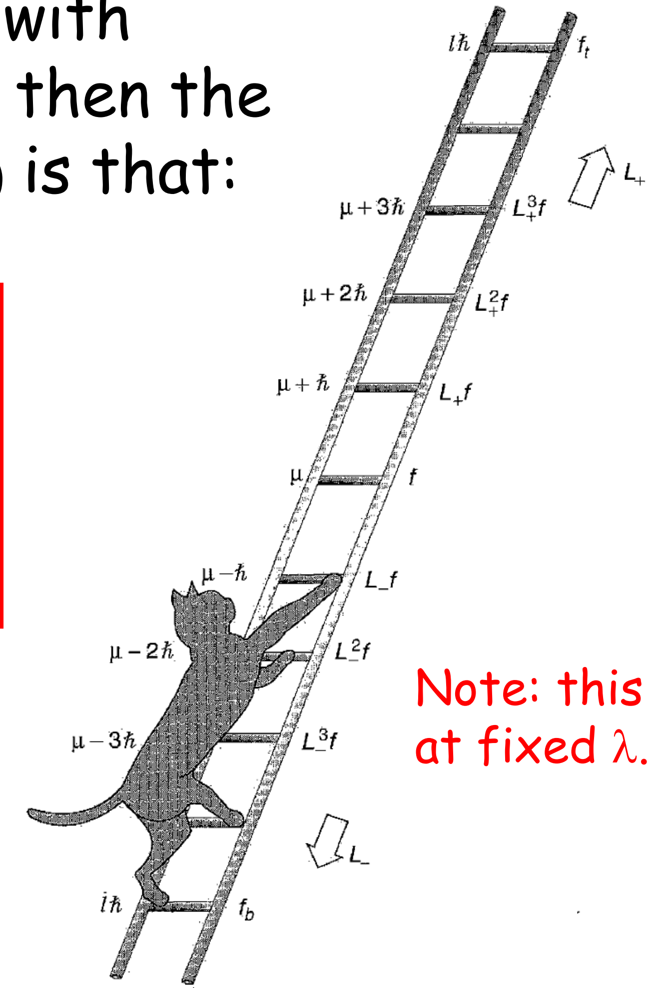




If f is eigenfunction of L^2 and L_z with eigenvalues λ and μ, respectively, then the claim (proof on page 158, similar as in Ch2) is that:

$$L^2(L_\pm f) = \lambda(L_\pm f)$$

$$L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$$



Note: this is at fixed λ.

L_+ is the raising operator and L_- the lowering operator.

But like in Ch.2 this cannot go on forever.
Eventually the projection, positive or negative, will be larger than the vector itself.

At the top value, let us call the L_z max eigenvalue $\hbar l$

$$L_z f_t = \hbar l f_t; \qquad L^2 f_t = \lambda f_t$$

Useful identity:

$$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x)$$

$$= L^2 - L_z^2 \mp i(i\hbar L_z)$$ Or, just reorganizing:

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

Two identities: use one.

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t =$$

$$= (0 + \hbar^2 l^2 + \hbar^2 l) f_t = \hbar^2 l(l+1) f_t$$

$$\lambda = \hbar^2 l(l+1)$$

On page 160, you can find the results starting at the bottom rung.

Too tedious to continue with all the details but you have the essence of the reasoning already. See page 160.

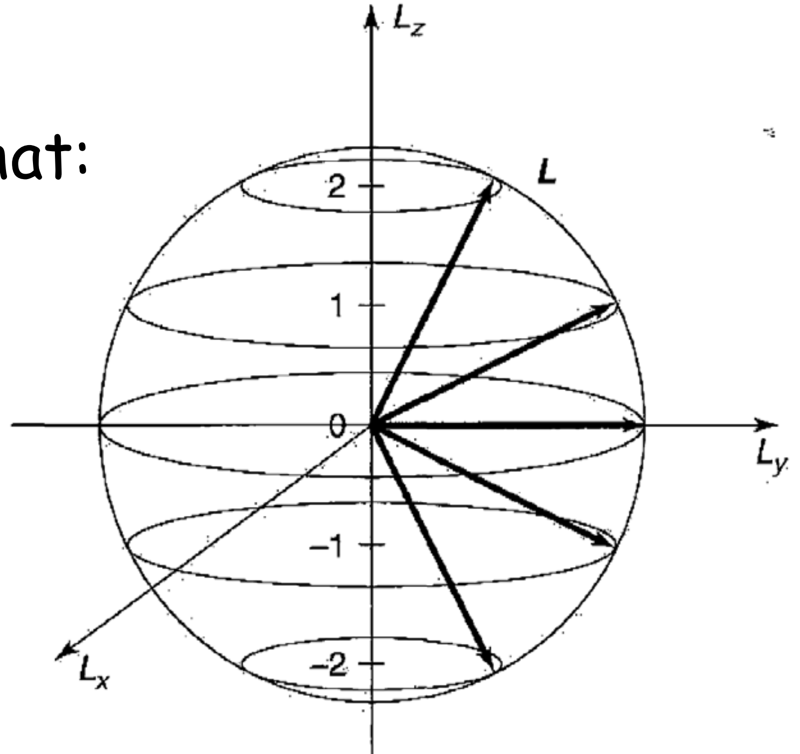
$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m$$

Analyzing now the bottom of the chain of states it can be shown that:

$$m = -l, \ -l+1, \ \ldots, \ l-1, \ l$$

thus $l = 0, \ 1/2, \ 1, \ 3/2, \ \ldots$

Note that *l* can be integer or half-integer mathematically speaking. For instance, l=3/2, can have m=3/2,1/2,-1/2,-3/2. All math satisfied! → Hint: spin!



Even the max projection has Lx and Ly uncertainty (read book)

In the previous page we found the meaning of "*l*" and "*m*" in the quantum numbers *(n,l,m)*.

The eigenvalues of the L^2 operator were $\hbar^2 l(l+1)$ and those of the L_z operator were $\hbar m$.

Now we need the eigenfunctions … (more difficult)

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

classical

$$\mathbf{p} \rightarrow \frac{\hbar}{i}\nabla$$

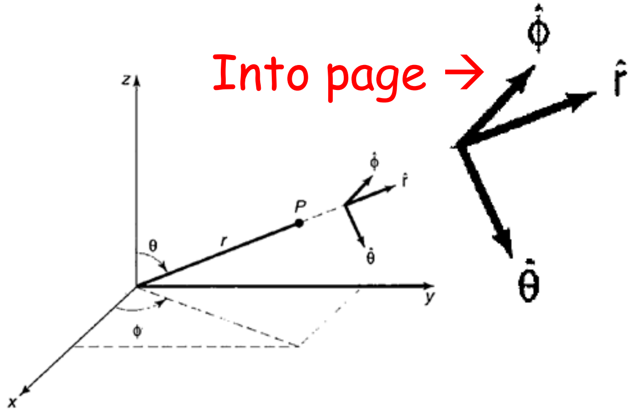
classical to QM

$$\mathbf{L} = (\hbar/i)(\mathbf{r} \times \nabla)$$

QM angular momentum

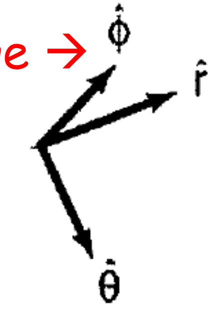
$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

(not obvious) gradient operator in spherical coordinates. Search in some math book.



Remember that $\mathbf{r} = r\hat{r}$. Then:

Into page →



$$\mathbf{L} = \frac{\hbar}{i}\left[r\underbrace{(\hat{r}\times\hat{r})}_{=0}\frac{\partial}{\partial r} + \underbrace{(\hat{r}\times\hat{\theta})}_{=\hat{\phi}}\frac{\partial}{\partial\theta} + \underbrace{(\hat{r}\times\hat{\phi})}_{=-\hat{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

We arrive to $\mathbf{L} = \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$

We can now rewrite in terms of the unit vectors in Cartesian coordinates using:

$$\hat{\theta} = (\cos\theta\cos\phi)\hat{\imath} + (\cos\theta\sin\phi)\hat{\jmath} - (\sin\theta)\hat{k}$$

$$\hat{\phi} = -(\sin\phi)\hat{\imath} + (\cos\phi)\hat{\jmath},$$

By mere replacement (easy) we arrive to:

$$L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_y = \frac{\hbar}{i}\left(+\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

Why do we care so much about x,y,z components instead of θ,ϕ,r components? Because we have all commutators etc etc written in terms of x,y,z components from previous pages …

The very important raising and lowering operators then become (again, easy):

$$L_{\pm} = L_x \pm i L_y \quad = \pm \hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi}\right)$$

Using a relation (easy) derived some pages back:

$$L^2 = L_{\pm} L_{\mp} + L_z^2 \mp \hbar L_z$$

we can deduce an expression (not as easy) for L^2 in spherical coordinates:

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

So we finally arrived to the differential equation we wish to solve to find the eigenfunctions:

$$L^2 f_l^m = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] f_l^m = \hbar^2 l(l+1) f_l^m$$

L^2 from previous page — from some pages back

HOWEVER, this equation happens to be the SAME "angular equation" that we derived at the start of Chapter 4 when we were trying "separation of variables" to solve the Sch. Eq. (see Eq.[4.17] book):

The "angular equation" was (a mere division by $-Y\hbar^2$ left and right is the only difference):

$$\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\}=-l(l+1)$$

We showed early in the chapter that the solutions were the Spherical Harmonics, so there is no further work to do!

Same for the other (much easier) equation (left as exercise):

$$L_z f_l^m = \frac{\hbar}{i}\frac{\partial}{\partial\phi} f_l^m = \hbar m f_l^m$$

In summary: the Spherical Harmonics, that we studied in detail before, are the eigenfunctions of the L^2 and L_z operators.

The eigenfunctions of the Hamiltonian of the Hydrogen atom

$$H\psi = E\psi, \quad L^2\psi = \hbar^2 l(l+1)\psi, \quad L_z\psi = \hbar m\psi$$

were already eigenfunctions of L^2 and L_z

Warning: we know spherical harmonics works well for integer *l*. For half-integer *l*, the story will be very different …

This completes the logic: the "*l*" and "*m*" quantum numbers introduced mathematically during the separation of variables procedure have a profound physical meaning related to rotations and angular momentum.

The eigenvalues of L^2 are $\hbar^2 l(l+1)$ and those of L_z are $\hbar m$.

Note that there is a $l(l+1)$ not a l^2. At large "*l*" the difference is small but at, say, $l=1$ it is not small.