4.4: Spin of electrons

A classical rigid body, as a planet, has two kinds of angular momenta: (1) ***L***, the orbital one associated with the center of mass, as Earth around the sun, and (2) ***S***, the spin, as Earth rotating daily about an axis.

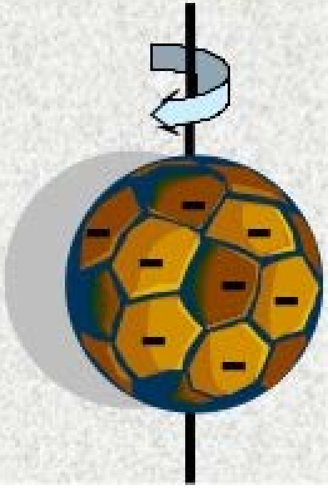
In quantum mechanics we already discussed the orbital component ***L*** (related with the electron around the nucleus).

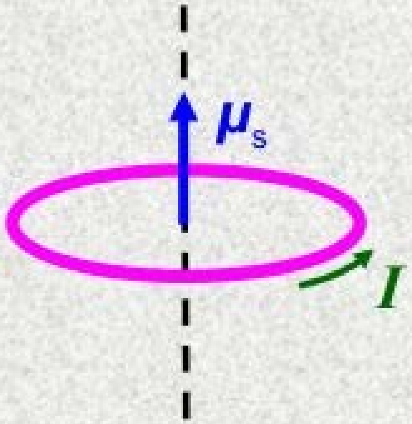
In QM, we also have a spin **S** for the electron but … the electron to the best of our accuracy is a POINT, thus should not rotate … in principle.

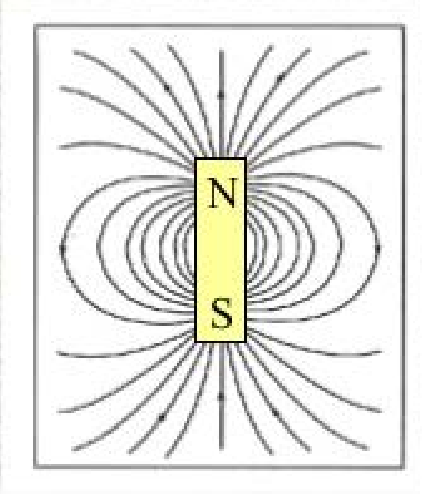
Spin Magnetic Dipole Moment

Just as electrons have the intrinsic properties of mass and charge, they have an intrinsic property called spin. This means that electrons, by their very nature, possess these three attributes. You're already comfortable with the notions of charge and mass. To understand spin it will be helpful to think of an electron as a rotating sphere or planet. However, this is no more than a helpful visual tool.

Imagine an electron as a soccer ball smeared with negative charge rotating about an axis. By the right hand rule, the angular momentum of the ball due to its rotation points down. But since its charge is negative, the spinning ball is like a little current loop flowing in the direction opposite its rotation, and the ball becomes an electromagnet with the N pole up. For an electron we would say its spin magnetic dipole moment vector, $\boldsymbol{\mu}_s$, points up. Because of its spin, an electron is like a little bar magnet.

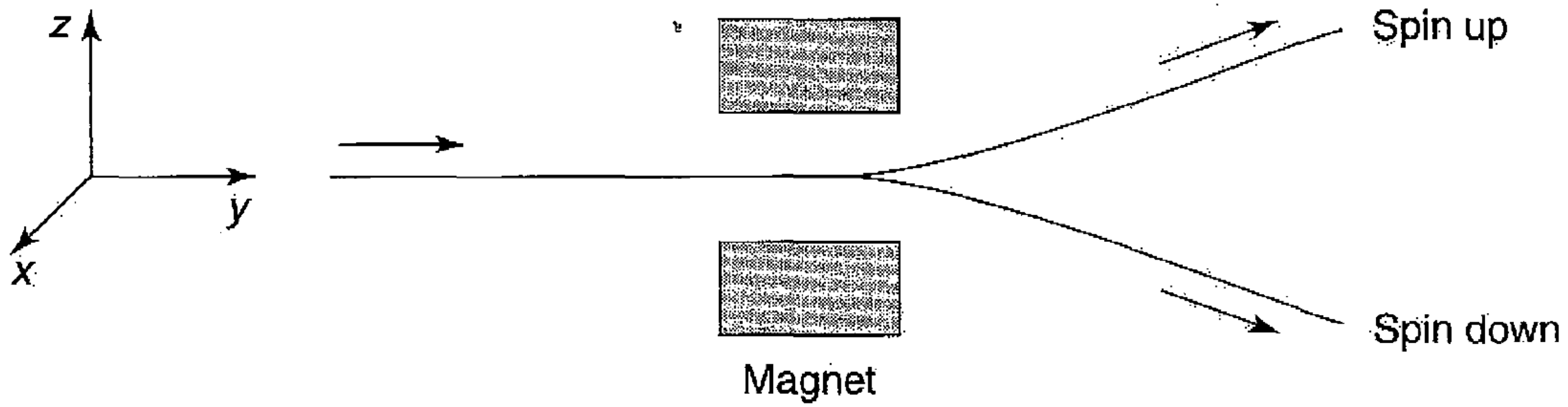


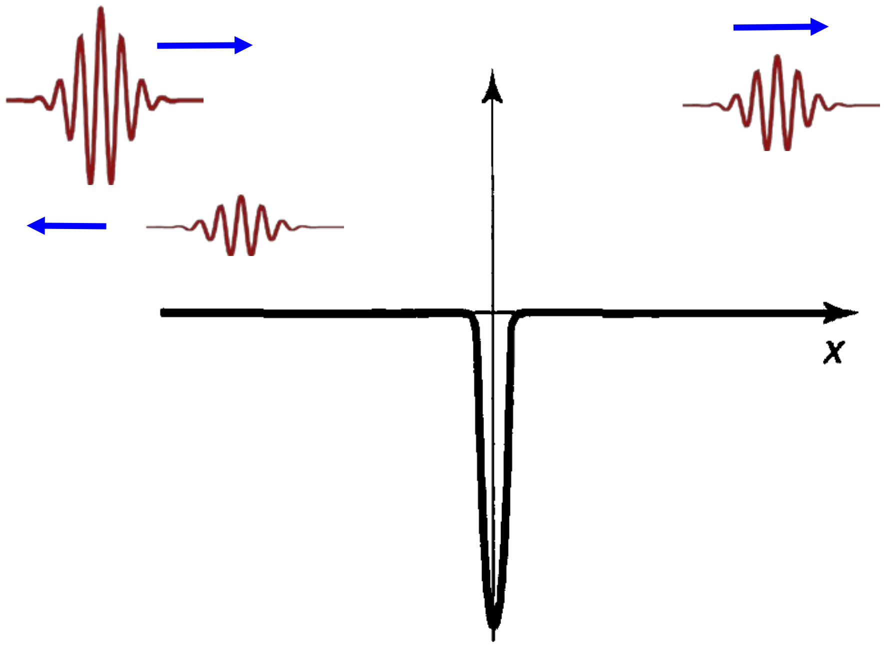




How do you know the electron has a spin? The same as for any magnetic moment, such as the orbital "l". You introduce the particle in a magnetic field and use the formula $E = -\mu.\mathbf{B}$ (note μ and $\mathbf{S}$ are opposite for an e^-)

In the 1921 Stern-Gerlach experiment a beam of electrons was used (actually a beam of silver atoms which according to the electron counting should have 1e^- in the outer 5s l=0 level).





However, once we measure the location and find the particle at position x_0, then that "sigma" width is gone.

The particle is exactly at x_0. At that moment, what radius it has?

In the wave packets shown, the "finite size" due to σ_x is the finite size of the probability wave function, not the particle itself.

Electron radius is smaller than 10^{-18} to 10^{-35} m according to experiments. Radius of nucleus is 10^{-15} m. Radius of atom is 10^{-10} m

Thus, it is a fact of Nature, that point elementary particles, such as an electron, carry an intrinsic spin angular momentum **S**.

Because the electron is a point, we cannot use the classical formula $\mathbf{S} = I\boldsymbol{\omega}$, i.e. no moment of inertia.

To describe the intrinsic spin, the math leads. It has to be "analogous" to that of *L*. Let us start with the commutators:

$$\boxed{[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y}$$

becomes the new set …

$$\boxed{[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y}$$

The eigenfunctions are more "abstract" ...

First, let us switch to the Ch. 3 notation using an abstract Hilbert space notation "$|\alpha>$" for states:

$$\boxed{L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m}$$ becomes

$$L^2 \,| l\, m_l > = \hbar^2\, l(l+1)\, | l\, m_l > ; \quad L_z \,| l\, m_l > = \hbar\, m_l\, | l\, m_l >$$

For L^2 and L_z using $Y_l^m(\theta, \phi)$ or $| l\, m_l >$ is the SAME.

But for the intrinsic spin, the Ch. 3 notation is the ONLY way.

Because in a previous lecture we arrived all the way to the eigenvalues by **only** using the commutators, then we simply repeat the operation line by line and find:

$$\boxed{S^2 \,|s\, m_s\rangle = \hbar^2\, s(s+1)\, |s\, m_s\rangle \,; \quad S_z \,|s\, m_s\rangle = \hbar\, m_s \,|s\, m_s\rangle}$$

$$s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \ldots\,; \quad m_s = -s,\ -s+1,\ \ldots,\ s-1,\ s$$

The spin of each type of particle is FIXED, not like the orbital angular momentum in *H* atom that you can change by emission or absorption of energy.

The projection of the spin of an electron can be changed, for instance by a flipping magnetic field, but the magnitude is intrinsic and fixed.

4.4.1: Spin ½ (electrons, quarks)

Use $S^2 \,|s\, m_s\rangle = \hbar^2\, s(s+1)\,|s\, m_s\rangle$; $S_z\,|s\, m_s\rangle = \hbar\, m_s\,|s\, m_s\rangle$

Specialize for s=1/2. Then, there are only two states, which in abstract form are:

$|\,\frac{1}{2}\;\frac{1}{2}\rangle$ and $|\,\frac{1}{2}\;-\frac{1}{2}\rangle$

We call them spin "up", or $\uparrow$, and spin "down", or $\downarrow$.

There is another, still abstract, way to represent spins up and down. It is using so-called "**spinors**"

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can combine the "up" and "down" linearly at will. So the spin could point "sideways" for instance.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_{+} + b\chi_{-}$$

If we use spinors for the states, then what do we use for the **operators** such as L^2 ? Certainly we cannot use derivatives of angles as for spherical harmonics.

From the two equations …

$$S^2 \,|\, \tfrac{1}{2} \;\; \tfrac{1}{2} > = \hbar^2 \, \tfrac{1}{2} \, (\tfrac{1}{2} + 1) \,|\, \tfrac{1}{2} \;\; \tfrac{1}{2} >$$

$$S^2 \,|\, \tfrac{1}{2} \; -\tfrac{1}{2} > = \underbrace{\hbar^2 \, \tfrac{1}{2} \, (\tfrac{1}{2} + 1)}_{\text{same}} \,|\, \tfrac{1}{2} \; -\tfrac{1}{2} >$$

… it can be deduced (see book, easy) that:

$$\boxed{\mathbf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

From the other two equations …

$S_z \mid \frac{1}{2}\ \frac{1}{2}> = \frac{1}{2}\,\hbar \mid \frac{1}{2}\ \frac{1}{2}>$

$S_z \mid \frac{1}{2}\ -\frac{1}{2}> = -\frac{1}{2}\,\hbar \mid \frac{1}{2}\ -\frac{1}{2}>$

… it can deduced (see book, easy) that:

$$\boxed{\mathbf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Using the S_+ and S_- operators [defined as $\mathbf{s}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{s}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$] it can be shown (book, page 168):

$$\boxed{\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

Dropping the ħ/2 factor defines
the famous **Pauli matrices**:

$$\boxed{\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Returning to the general combination:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

It has to be normalized like any other state i.e.

$$|a|^2 + |b|^2 = 1$$

$|a|^2$ is the probability of measuring spin up.

$|b|^2$ is the probability of measuring spin down.