Consider now scattering states with $E>0$ (page 66). If x nonzero, then the Sch. Eq. is the same as for free particles.

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \qquad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad x<0$$
$$\psi(x) = Fe^{ikx} + Ge^{-ikx} \quad x>0$$

Continuity at x=0:

$$\boxed{F + G = A + B}$$

$$d\psi/dx = ik\left(Fe^{ikx} - Ge^{-ikx}\right) \quad x>0$$
$$d\psi/dx = ik\left(Ae^{ikx} - Be^{-ikx}\right) \quad x<0$$

Remember here the derivative is discontinuous at x=0 because of the exotic nature of V(x).

From the same bound state analysis done before we find:

$$-\frac{\hbar^2}{2m}\underbrace{\left(\left.\frac{d\psi}{dx}\right|_{+\epsilon}-\left.\frac{d\psi}{dx}\right|_{-\epsilon}\right)}_{ik(F-G-A+B)} - \alpha \underbrace{\psi(0)}_{A+B} = 0$$

$$ik(F-G-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

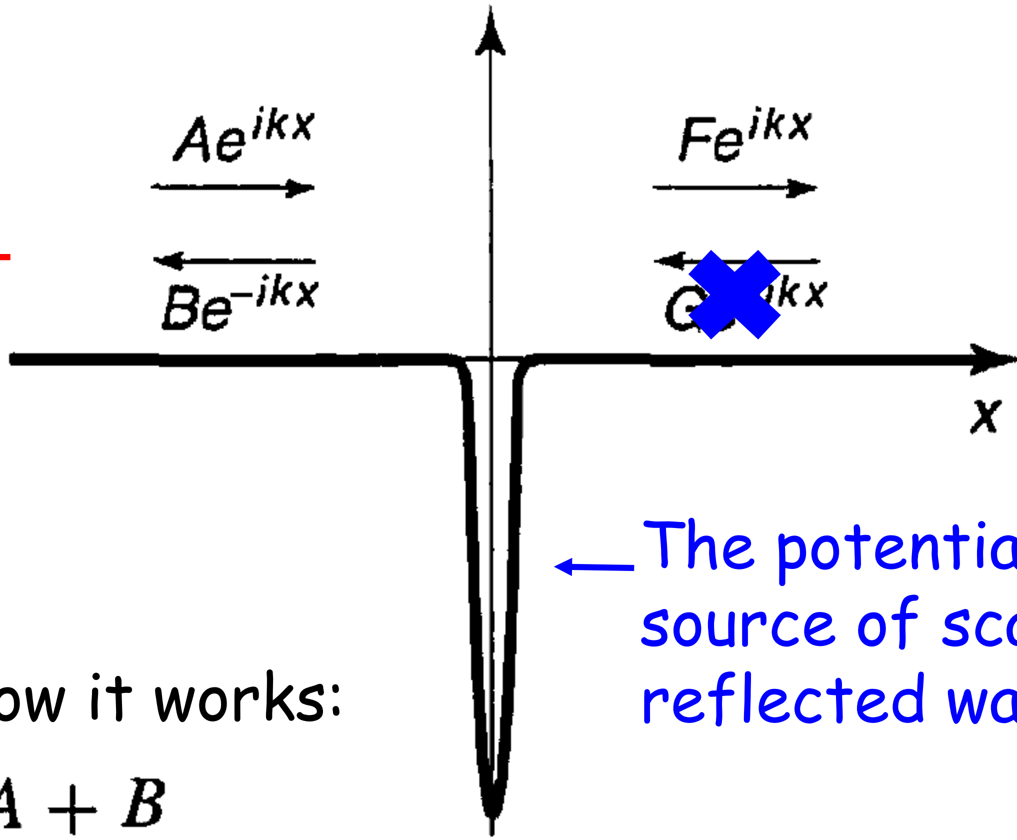
Four unknowns and two equations (framed in red). Something missing …

These are not normalizable states so "strange" behavior is expected. We need to think "physically" what we are doing in terms of a real scattering experiment.

Real scattering experiment:

Source →

Detector ← will measure reflected e-



→ Detector will measure transmitted e-

← The potential at x=0 is a source of scattering, and reflected waves.

Example of how it works:

$$F + \cancel{G} = A + B$$

Divide by A both eqs:

$$F/A = 1 + B/A$$

Now only two unknowns!

$$ik(F - \cancel{G} - A + B) = -\frac{2m\alpha}{\hbar^2}(A + B)$$

Divide by A and again only B/A and F/A are unknowns.

Introducing $\beta \equiv \frac{m\alpha}{\hbar^2 k}$ it can be shown that:

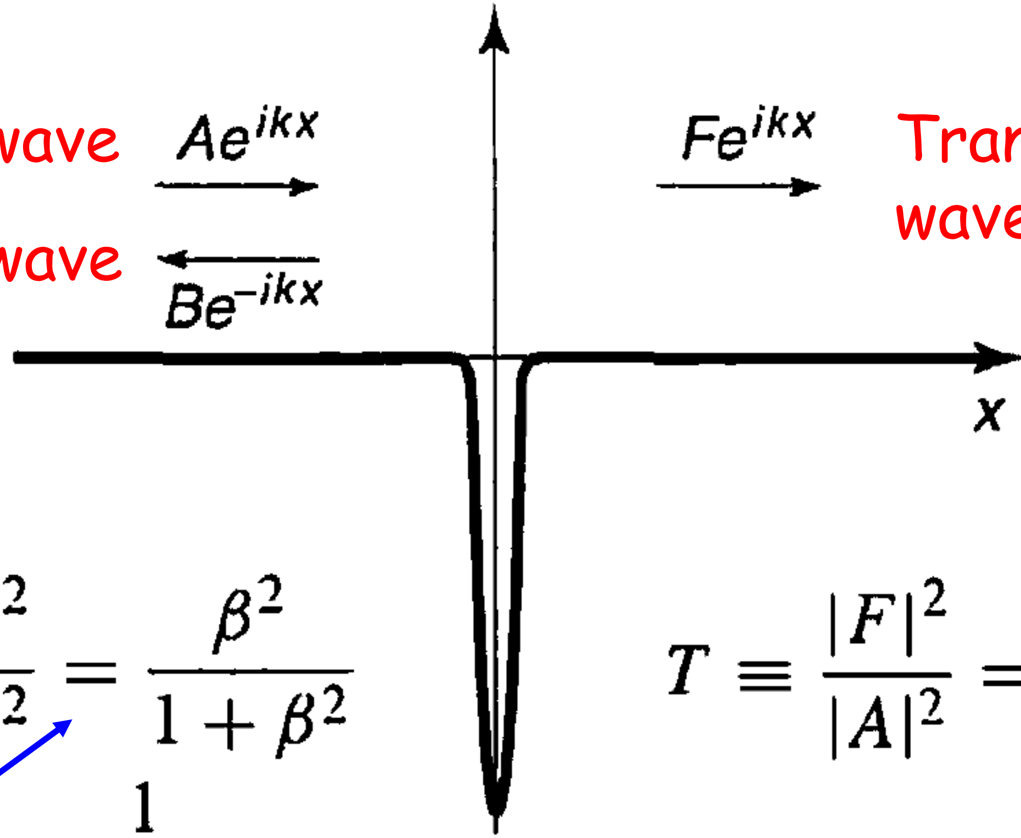
$$B = \frac{i\beta}{1 - i\beta}A, \quad F = \frac{1}{1 - i\beta}A$$

Because probabilities are related to $|\psi|^2$ what matters are:

$$R \equiv \frac{|B|^2}{|A|^2} \qquad T \equiv \frac{|F|^2}{|A|^2}$$

They should satisfy: $R + T = 1$

Summary scattering experiment (any E>0):



$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$$

$$= \frac{1}{1+(2\hbar^2 E/m\alpha^2)}$$

δ potential

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

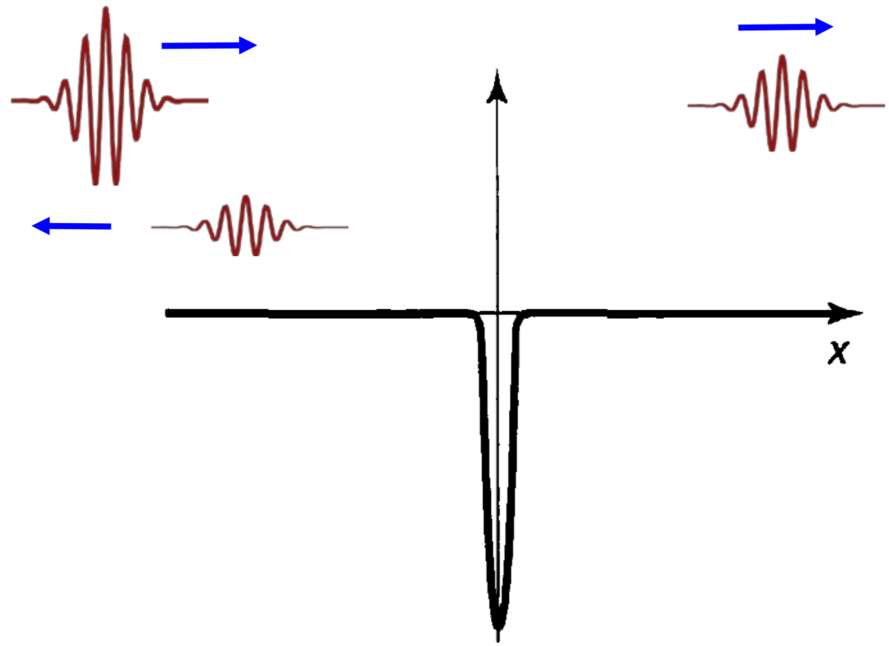
δ potential

$$= \frac{1}{1+(m\alpha^2/2\hbar^2 E)}$$

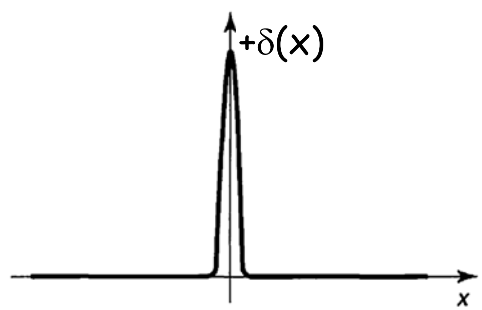
$$R + T = 1$$ √

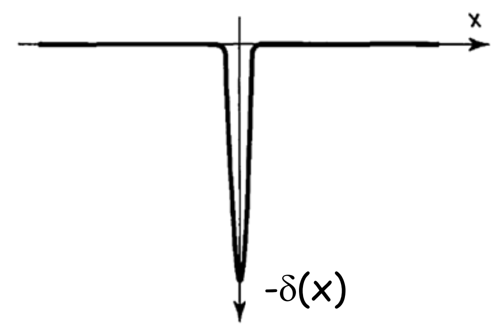
Two interesting comments:

(1) We used not normalizable solutions, but we meant to use **wave packets**:

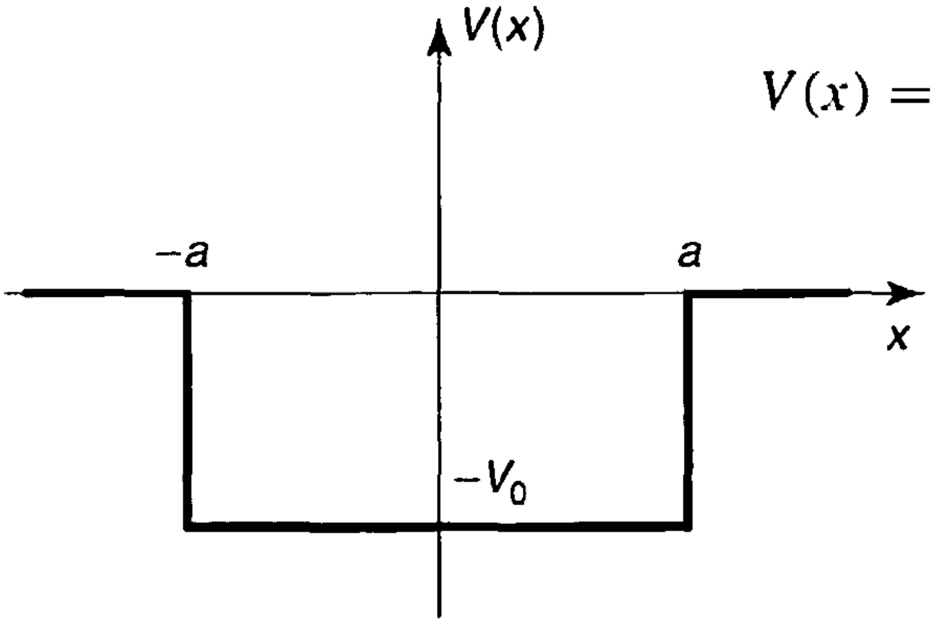


(2) For the scattering problem the sign of α does not matter !!
Repulsive or attractive is the same for scattering [but not for bound state which only occurs for $-\delta(x)$].





2.6 The finite square well



$$V(x) = \begin{cases} -V_0, & \text{for } -a \le x \le a, \\ 0, & \text{for } |x| > a, \end{cases}$$

We anticipate it will have both bound and scattering states.

PROCEDURE: There are three regions. Crucially, I can solve the equation in each! Thus, we will propose a general solution in each, and then match ψ and $d\psi/dx$ at the two boundaries.

Let us start with bound states i.e. $E<0$.

Left region x<-a and **Right** region x>a:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \kappa^2\psi \qquad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi(x) = Be^{\kappa x}, \quad \text{for } x < -a.$$

$$\psi(x) = Fe^{-\kappa x}. \quad \text{for } x > a.$$

The "other" exponential diverges in each case.

Middle region -a<x<a. Here E>-V_0

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -l^2\psi$$

$$l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar} > 0$$

In middle region, the general solution is:

$$\psi(x) = C\sin(lx) + D\cos(lx)$$

I cannot drop any term a priori.

Note: I could have proposed a sum of e^{ilx} and e^{-ilx} with two unknowns as well.

Five unknowns *A,C,D,F* (plus *E*) but I have five eqs: match of ψ and $d\psi/dx$ at x=a and x=-a, and normalization.

Moreover, the solutions must be even or odd under x → -x. *I can study each sector separately.*

We will do the even sector (the odd sector will be in HW).
Only F and D are unknowns.

even

$$\psi(x) = \begin{cases} F e^{-\kappa x}, & \text{for } x > a, \\ D\cos(lx), & \text{for } -a < x < a, \\ \psi(-x), & \text{for } x < -a \end{cases}$$

Continuity of ψ at x=a: $F e^{-\kappa a} = D\cos(la)$

Continuity of dψ/dx at x=a: $-\kappa F e^{-\kappa a} = -l D \sin(la)$

From ratio we get $\boxed{\kappa = l\tan(la)}$ where

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \quad l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

This equation cannot be solved exactly. Must be done numerically. *In general this is the most common situation.*

Also in general there is no need to use all the numerical values of masses, Planck constant, etc. Use clever variables.

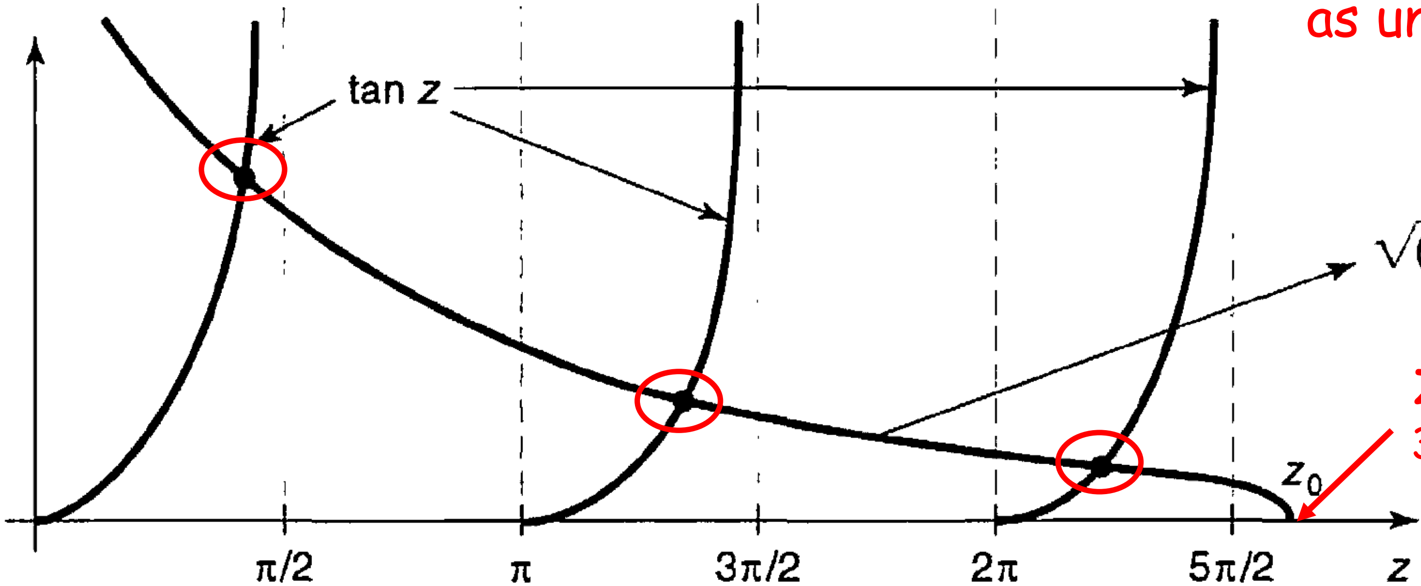
Left as exercise

$$z \equiv la \qquad z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0} \;\rightarrow\; \kappa a = \sqrt{z_0^2 - z^2}$$

Dimensionless combo

$$\rightarrow \qquad \tan z = \sqrt{(z_0/z)^2 - 1}$$

We trade E as unknown to z as unknown.



z_0=8
3 solutions