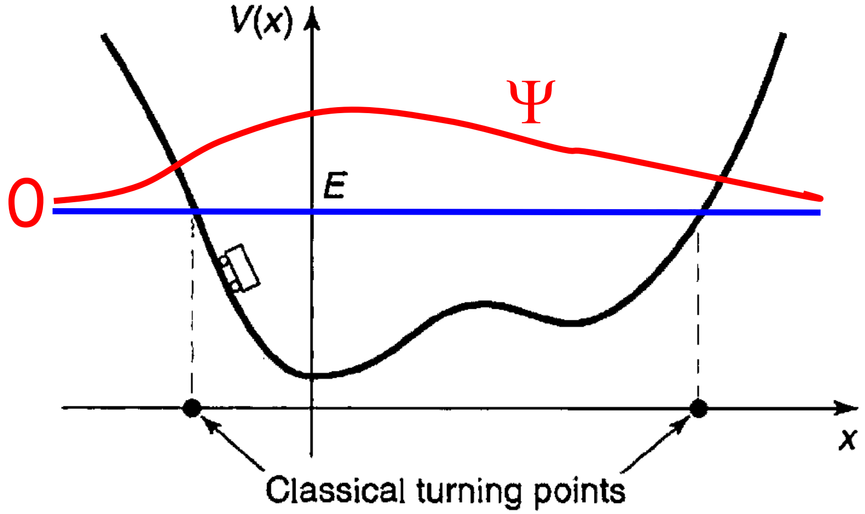
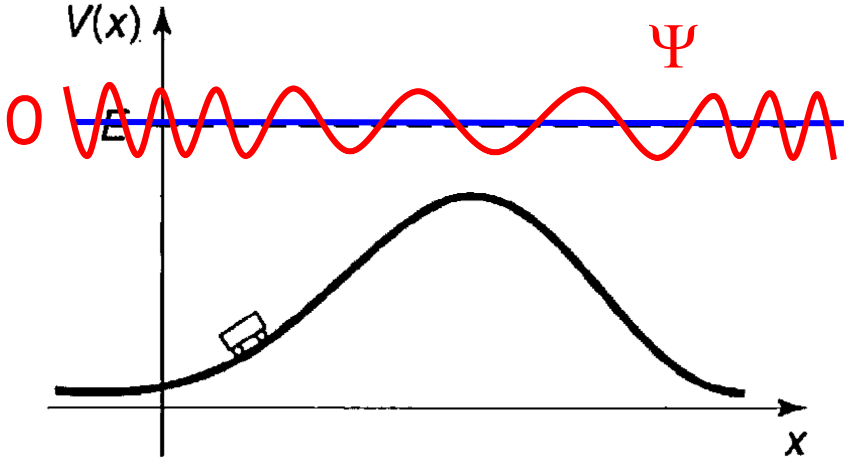
2.5.1 Bound vs scattering states

In V(x) below, with any E < V(+-infinity), classically the particle oscillates back and forth. Cannot escape. Any E is good.



Bound state

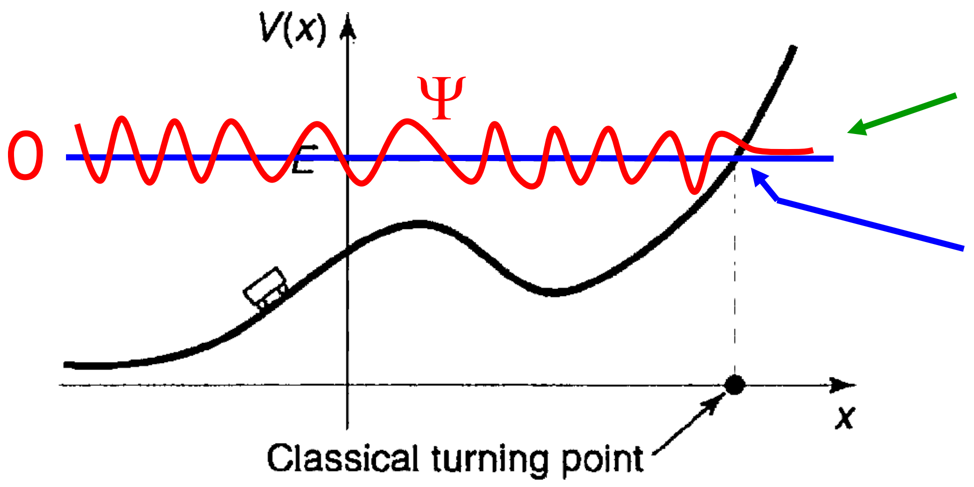
In QM, however, E is discrete (index n) and Ψ is normalizable. Particle still cannot escape because $\Psi \rightarrow$ +- infinite.



Here, E is a "scattering state" both in QM and classically, because $E>V(+-$ infinity).

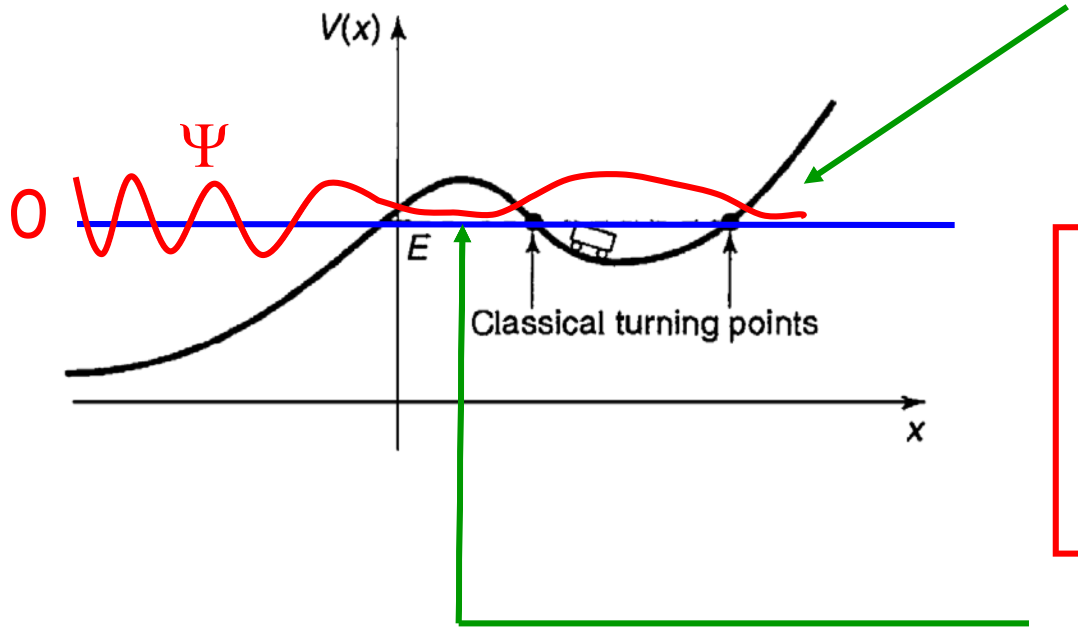
In **both**, E is continuous. In QM, solutions are not normalizable, but wave packets save the day.

Students must develop the ability to sketch the wave function even without solving the problem (most problems cannot be solved exactly). Remember Test 1.



Scattering states exist both in QM and classically, for this E and $V(x)$.

In QM, there is a small penetration beyond classical point.



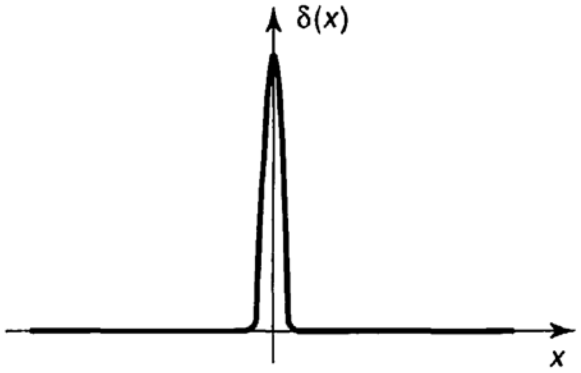
Classically this E can be either bound or scattering state depending on initial location at t=0.

In QM, it is **only one** scattering state.
Particle can escape to the left!

Reason: tunneling.

2.5.2 The delta function well (or potential)

The Dirac delta function represents a localized heavy object like a neutron that electrons may collide with.



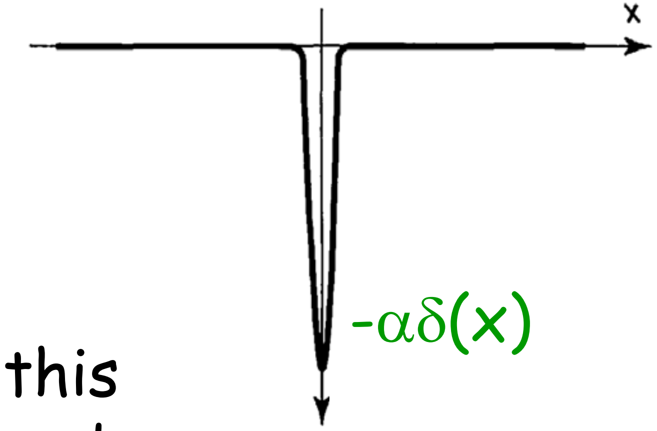
$$\delta(x) \equiv \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}, \quad \text{with} \int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\, dx = f(a) \int_{-\infty}^{+\infty} \delta(x-a)\, dx = f(a)$$

The Dirac delta function can be positive (repulsive, as shown) or negative (attractive).

Consider a potential $V(x) = -\alpha\delta(x)$ (α>0) The Sch. Eq. is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi = E\psi$$



Because potential at +- infinity is 0, this potential can have both bound (E<0) and scattering (E>0) states.

If x is nonzero, and E<0 to explore bound states, the Sch. Eq. is "almost" the same as for free particle:

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = \kappa^2\psi \qquad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

Sign difference with free particle

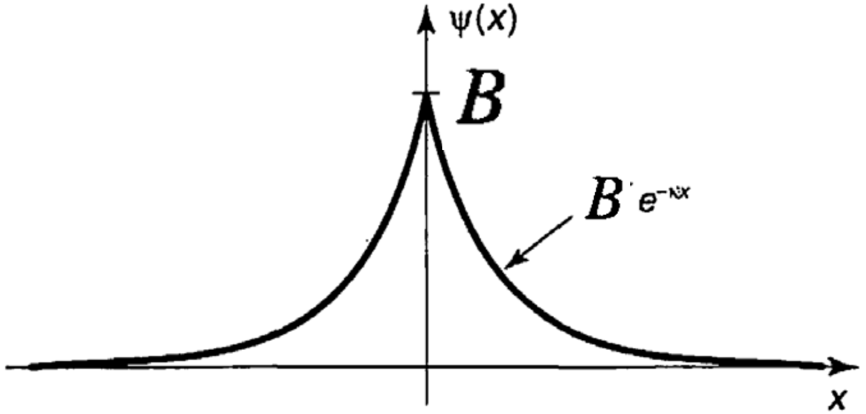
Because $\kappa^2>0$, then the general solution is:

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x}$$ Check

For x<0, A=0 otherwise diverges. $$\psi(x) = Be^{\kappa x} \quad (x < 0)$$

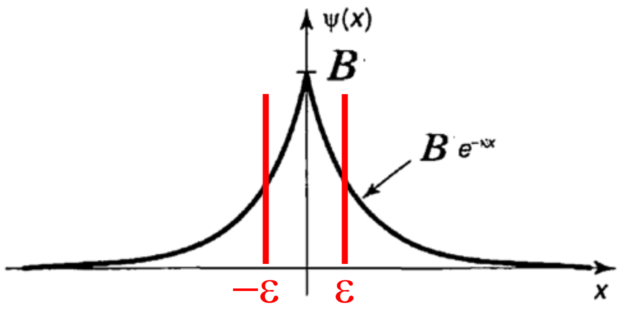
For x>0, similar reasoning leads to: $$\psi(x) = Fe^{-\kappa x} \quad (x > 0)$$

By continuity at x=0, then B=F.



We have the shape of $\Psi(x)$ but not E yet, plus we have not used α.

How do we find E? In dealing with δ-functions, often we need to integrate near the δ-function. Use this "trick" for Sch. Eq., send epsilon -> 0 at the end.



$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}\,dx+\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx=E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dx}\right|_{+\epsilon}-\left.\frac{d\psi}{dx}\right|_{-\epsilon}\right)$$

Easy, since I have Ψ already.

$-\alpha\ \psi(0)$

B (=F)

0 because Ψ continuous

$$\begin{cases} d\psi/dx = -B\kappa e^{-\kappa x}, & \text{for } (x > 0), \quad \text{so } d\psi/dx\big|_{+} = -B\kappa, \\ d\psi/dx = +B\kappa e^{+\kappa x}, & \text{for } (x < 0), \quad \text{so } d\psi/dx\big|_{-} = +B\kappa. \end{cases}$$

Then, integrated Sch. Eq. becomes $-\frac{\hbar^2}{2m}(-2\cancel{B}\kappa) - \alpha \cancel{B} = 0$

Note: *B* cancels and ε does not appear.

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

(1) *B* arises from normalization, left as exercise (check).

(2) Only because *V(x)* diverges, $d\Psi/dx$ is discontinuous.

Otherwise it is continuous (i.e. second term, previous page, gives 0)

The (only) bound state and its energy is:

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \qquad E = -\frac{m\alpha^2}{2\hbar^2}.$$