Consider now scattering states with $E>0$ (page 66). If x nonzero, then the Sch. Eq. is the same as for free particles.

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \qquad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad x<0$$

$$\psi(x) = Fe^{ikx} + Ge^{-ikx} \quad x>0$$

Continuity at x=0:

$$\boxed{F + G = A + B}$$

$$d\psi/dx = ik\left(Fe^{ikx} - Ge^{-ikx}\right) \quad x>0$$

$$d\psi/dx = ik\left(Ae^{ikx} - Be^{-ikx}\right) \quad x<0$$

Remember here the derivative is discontinuous at x=0 because of the exotic nature of V(x).

From the same bound state analysis done before we find:

$$-\frac{\hbar^2}{2m}\left(\underbrace{\left.\frac{d\psi}{dx}\right|_{+\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon}}_{ik(F-G-A+B)}\right) - \alpha\, \underbrace{\psi(0)}_{A+B} = 0$$

$$ik(F - G - A + B) = -\frac{2m\alpha}{\hbar^2}(A + B)$$

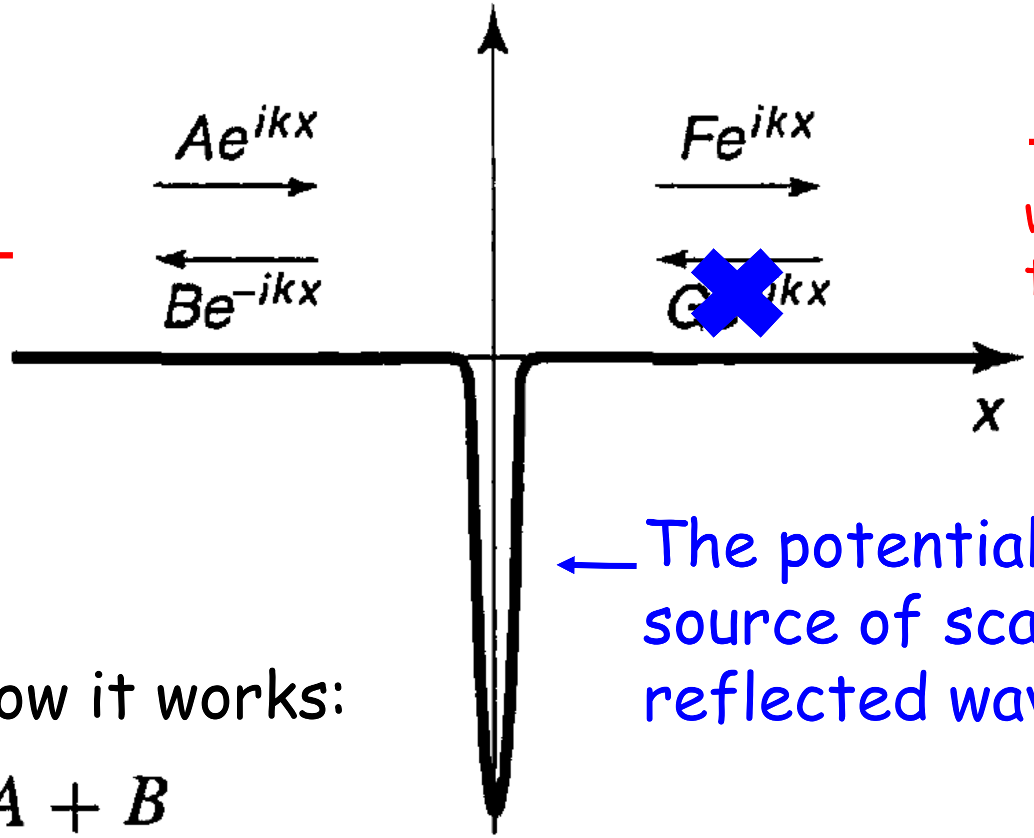
Four unknowns and two equations. Something missing …

These are not normalizable states so "strange" behavior is expected. We need to think "physically" what we are doing in terms of a real scattering experiment.

Real scattering experiment:

Source →

Detector ← will measure reflected e-



→ Detector will measure transmitted e-

← The potential at x=0 is a source of scattering, and reflected waves.

Example of how it works:

$$F + \cancel{G} = A + B$$

Divide by A both eqs:

$$F/A = 1 + B/A$$

Now only two unknowns!

$$ik(F - \cancel{G} - A + B) = -\frac{2m\alpha}{\hbar^2}(A + B)$$

Divide by *A* and again only *B/A* and *F/A* are unknowns.

Introducing $\beta \equiv \frac{m\alpha}{\hbar^2 k}$ it can be shown that:

$$B = \frac{i\beta}{1 - i\beta}A, \quad F = \frac{1}{1 - i\beta}A$$

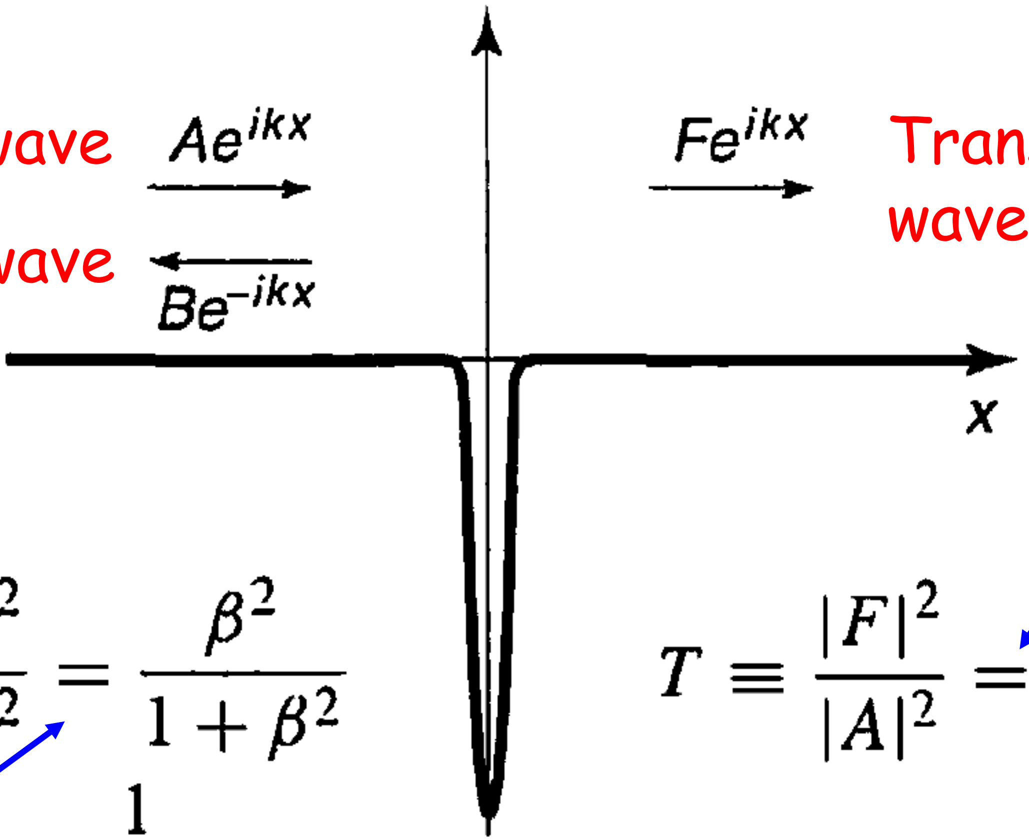
Check!

Because probabilities are related to $|\psi|^2$ what matters are:

$$R \equiv \frac{|B|^2}{|A|^2} \qquad T \equiv \frac{|F|^2}{|A|^2}$$

They should satisfy: $R + T = 1$

Summary scattering experiment (any E>0):



$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} = \frac{1}{1+(2\hbar^2 E/m\alpha^2)}$$

δ potential

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2} = \frac{1}{1+(m\alpha^2/2\hbar^2 E)}$$

δ potential

$$R + T = 1 \checkmark$$

$$ik(F - G - A + B) = -\frac{2m\alpha}{\hbar^2}(A + B)$$

Divide by A and again only B/A and F/A are unknowns.

Introducing $\beta \equiv \frac{m\alpha}{\hbar^2 k}$ it can be shown that:

$$B = \frac{i\beta}{1 - i\beta}A, \quad F = \frac{1}{1 - i\beta}A$$

Check!

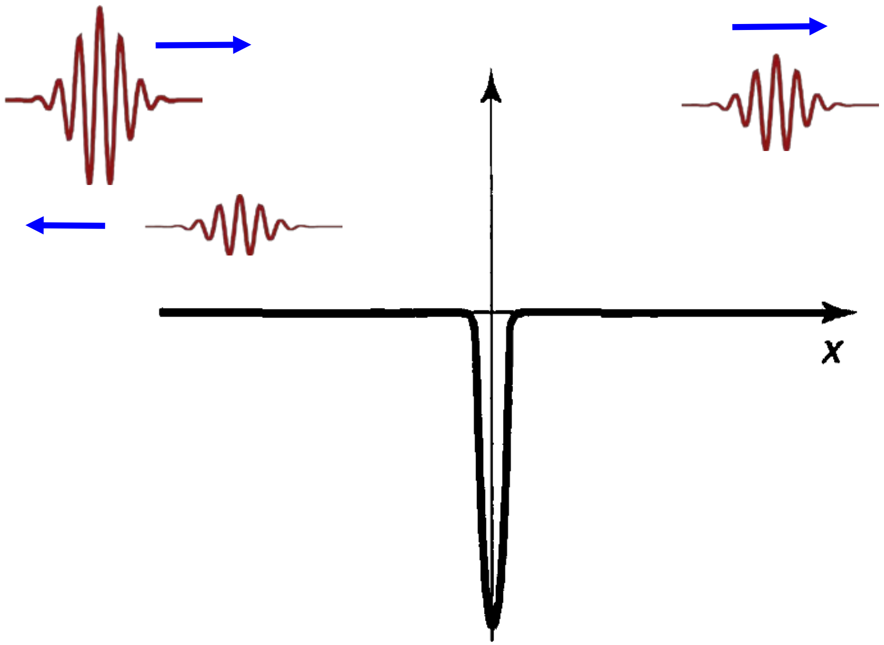
Because probabilities are related to $|\psi|^2$ what matters are:

$$R \equiv \frac{|B|^2}{|A|^2} \qquad T \equiv \frac{|F|^2}{|A|^2}$$

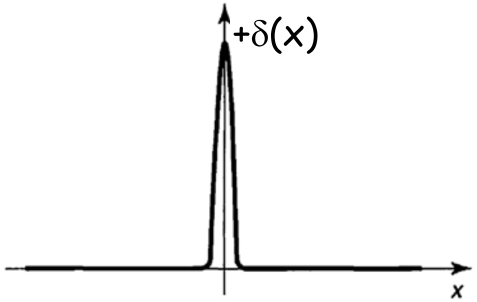
They should satisfy: $R + T = 1$.

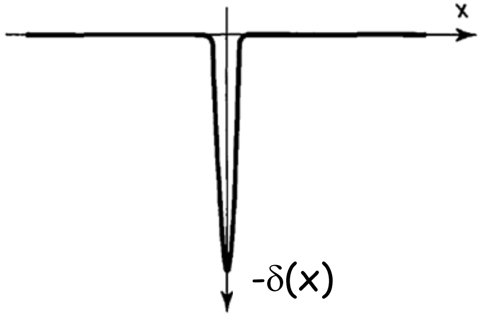
Two interesting comments:

(1) We used not normalizable solutions, but we meant to use **wave packets**:

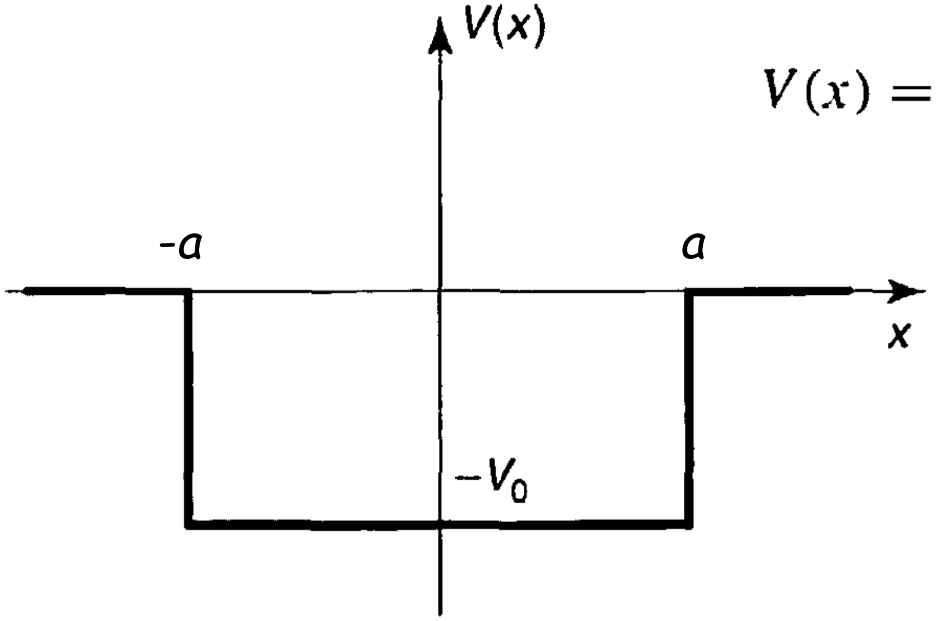


(2) For the scattering problem the sign of α does not matter ! Repulsive or attractive is the same for scattering [but not for bound state which only occurs for $-\delta(x)$].





2.6 The finite square well



$$V(x) = \begin{cases} -V_0, & \text{for } -a \le x \le a, \\ 0, & \text{for } |x| > a, \end{cases}$$

We anticipate it will have both bound and scattering states.

PROCEDURE: There are three regions (piecewise-defined function). We will propose a general solution in each, and then match ψ and $d\psi/dx$ at the two boundaries.

No weird "tricks" needed for $d\psi/dx$, as in delta function.

Let us start with bound states i.e. $E<0$.

Left region $x < -a$ and **Right** region $x > a$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \kappa^2\psi \quad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi(x) = Be^{\kappa x}, \quad \text{for } x < -a.$$

$$\psi(x) = Fe^{-\kappa x}. \quad \text{for } x > a.$$

The "other" exponential diverges in each case. Exactly the same as with delta function.

Middle region $-a<x<a$. Here $E>-V_0$ (like say -5 > -7)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -l^2\psi$$

$$l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar} > 0$$