There are many theorems for Hermitian operators, the operators that matter for observables, that I will NOT prove:

(1) The eigenvalues q are real, like the energies E_n were. If you measure $\hat{Q}$ in any $\Psi(x,t)$, you will get one of the q's.

(2) The eigenfunctions $f_q(x)$ for different q's are orthonormal -- sometimes using Kronecker delta, sometimes Dirac delta -- like $\Psi_n(x)$ for energies were.

(3) The eigenfunctions are complete, like $\Psi_n(x)$ for energies were.

Caveat: careful with degenerate states i.e. those with the same eigenvalue q.

Generalized statistical interpretation

Suppose the electron is in a state $\Psi(x,t)$. We know that the probability of measuring E_n is $|c_n|^2 = |<\Psi_n(x)| \Psi(x,t)>|^2$.

Suppose in the **same** state $\Psi(x,t)$ I measure say the momentum, or angular momentum, etc. What will I find?

If I measure the Hermitian (observable) operator $\hat{M}$, the probability of finding "m" is $|c_m|^2 = |< f_m(x) | \Psi(x,t) >|^2$ if the eigenvalues are discrete, like in angular momentum.

Like with the energy, the total probability of measuring "some" value for operator $\hat{M}$ must be 1.

$$\Sigma_m |c_m|^2 = 1$$

For eigenvalues that are continuous, the probability of measuring "p", like a linear momentum, requires a tiny width for its definition

$$|c(p)|^2 \, dp = | \langle f_p(x) \mid \Psi(x,t) \rangle |^2 \, dp,$$

dimensionless

and then you must integrate in a finite range from say p_a to p_b.

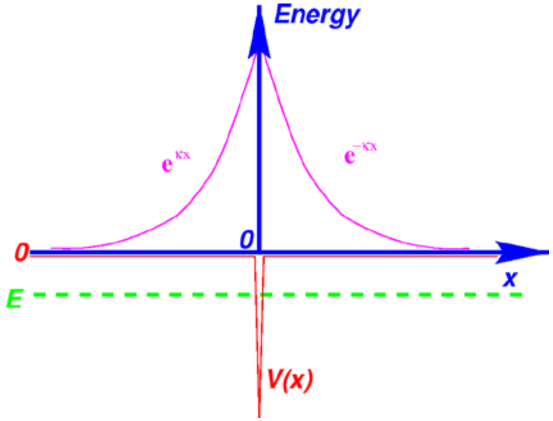
For the operator $\hat{x}$, we recover the old result (see book page 108; warning a bit complicated):

$$\int_a^b |\Psi(x,t)|^2 \, dx = \left\{ \begin{array}{l} \text{probability of finding the particle} \\ \text{between } a \text{ and } b \text{, at time } t. \end{array} \right\}$$

Example 3.4 book:

Consider a particle located in the (only) bound state of the δ-function potential. The wave function is:

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} e^{-iEt/\hbar}$$



Typical question: what is the probability of measuring a momentum greater than $p_0 = m\alpha/\hbar^*$? We need to calculate $|c(p)|^2$ (see next page).

* Can you confirm that the units are those of momentum?

Reminder: eigenfunction of momentum operator $\hat{p}$. We need an eigenfunction $f_p(x)$ such that

eigenfunction

$\hat{p}$ operator

$$\frac{\hbar}{i}\frac{d}{dx}f_p(x) = p\,f_p(x)$$

eigenvalue

Solution is very easy (but normalization is complicated):

$$\boxed{f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}}$$

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} e^{-iEt/\hbar}$$

Repeating: What is the probability of measuring a momentum greater than $p_0 = m\alpha/\hbar$?

We need to calculate $|c(p)|^2$.

$$c(p) = \langle f_p|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx \quad =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{m\alpha}}{\hbar} e^{-iEt/\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-m\alpha|x|/\hbar^2}\, dx = \boxed{\sqrt{\frac{2}{\pi}}\, \frac{p_0^{3/2} e^{-iEt/\hbar}}{p^2 + p_0^2}}$$

integral given to students

The final answer is:

Integral given. Answer must be a number in the range [0,1]

$$\int_{p_0}^{\infty} |c(p)|^2 \, dp = \frac{2}{\pi} p_0^3 \int_{p_0}^{\infty} \frac{1}{(p^2 + p_0^2)^2} \, dp = 0.0908$$

Just a name: c(p), which can be function of t but not x, is often called Φ(p,t), the momentum space wave function.

A similar problem could be formulated for the harmonic oscillator involving Gaussians (HW problem) or the infinite square well involving sines, etc., etc.

Uncertainty Principle *(3 pages, prepare for impact)*

The standard deviation for any operator is
(note $<\Psi|\hat{Q}\Psi> = <\Psi|\hat{Q}|\Psi>$, i.e. $<\Psi|\hat{x}\Psi> = <\Psi|\hat{x}|\Psi>$,).

$$\sigma_A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = \langle \Psi | \hat{A}^2 | \Psi \rangle - \langle \Psi | \langle \hat{A} \rangle \hat{A} | \Psi \rangle =$$

$$= \langle \Psi | \hat{A}^2 - 2 \langle \hat{A} \rangle \hat{A} + \langle \hat{A} \rangle \hat{A} | \Psi \rangle = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \Psi \rangle$$

If Ψ normalized to 1
i.e. $< \Psi \mid \Psi > = 1$.

For Hermitian operator this can be rewritten as:

For $\hat{A}$: $\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)\Psi | (\hat{A} - \langle \hat{A} \rangle)\Psi \rangle = \langle f | f \rangle$ where

$$f \equiv (\hat{A} - \langle \hat{A} \rangle)\Psi$$

For $\hat{B}$: $\sigma_B^2 = \langle g | g \rangle$ $\quad g \equiv (\hat{B} - \langle \hat{B} \rangle)\Psi$

Consider the Schwartz inequality:

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\underbrace{\langle f|g\rangle}_{z}|^2$$

We use now the following property of complex numbers:

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

Consider $z = \langle f|g\rangle$

Then .... $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2$

Then (repeated) …. $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2$

$\hat{A}$ is Hermitian

$$\langle f|g\rangle = \langle(\hat{A} - \langle\hat{A}\rangle)\Psi|(\hat{B} - \langle\hat{B}\rangle)\Psi\rangle = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle)\Psi\rangle$$

$$= \langle\Psi|(\hat{A}\hat{B} - \hat{A}\langle\hat{B}\rangle - \hat{B}\langle\hat{A}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle)\Psi\rangle$$

$$= \langle\Psi|\hat{A}\hat{B}\Psi\rangle - \langle\hat{B}\rangle\langle\Psi|\hat{A}\Psi\rangle - \langle\hat{A}\rangle\langle\Psi|\hat{B}\Psi\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle\langle\Psi|\Psi\rangle$$

$$= \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\rangle\langle\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$= \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle.$$

Make sure you understand every step

$$\langle g|f\rangle = \langle\hat{B}\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

Left as exercise

In summary: $\langle f|g\rangle - \langle g|f\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle = \langle[\hat{A}, \hat{B}]\rangle$

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Generalized uncertainty principle:

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

Assumes <..> is in a normalized to 1 state, and both operators Hermitian.

As special case, if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, then $[\hat{x}, \hat{p}] = i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2 = \left(\frac{\hbar}{2} \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

It can also be trivial: if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{x}^2$, then $\sigma_x \sigma_{x2} = 0$

This was the last item of Ch. 3 for us