In middle region, the general solution is:

$$\psi(x) = C\sin(lx) + D\cos(lx)$$

I cannot drop any term a priori.

Note: I could have proposed a sum of e^{ilx} and e^{-ilx} with two unknowns as well.

Five unknowns *A,C,D,F* (plus *E*) and I have five eqs: match of ψ and $d\psi/dx$ at x=a and x=-a, and normalization.

Moreover, the solutions must be even or odd under x → -x. *I can study each sector separately.*

We will do the even sector (odd sector will be in HW).Only F and D are unknowns. We only need to study $x=a$ ($x=-a$ will give same info)

even

$$\psi(x) = \begin{cases} Fe^{-\kappa x}, & \text{for } x > a, \\ D\cos(lx), & \text{for } -a < x < a, \\ \psi(-x), & \text{for } x < -a \end{cases}$$

even

For odd I would use the sine function in the middle and for $x<-a$, $-\psi(-x)$

Continuity of ψ at $x=a$: $$Fe^{-\kappa a} = D\cos(la)$$

Continuity of $d\psi/dx$ at $x=a$: $$-\kappa Fe^{-\kappa a} = -lD\sin(la)$$

From ratio we get $\boxed{\kappa = l\tan(la)}$ where

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \qquad l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

This equation cannot be solved exactly. Must be done numerically. *In general this is the most common situation.*

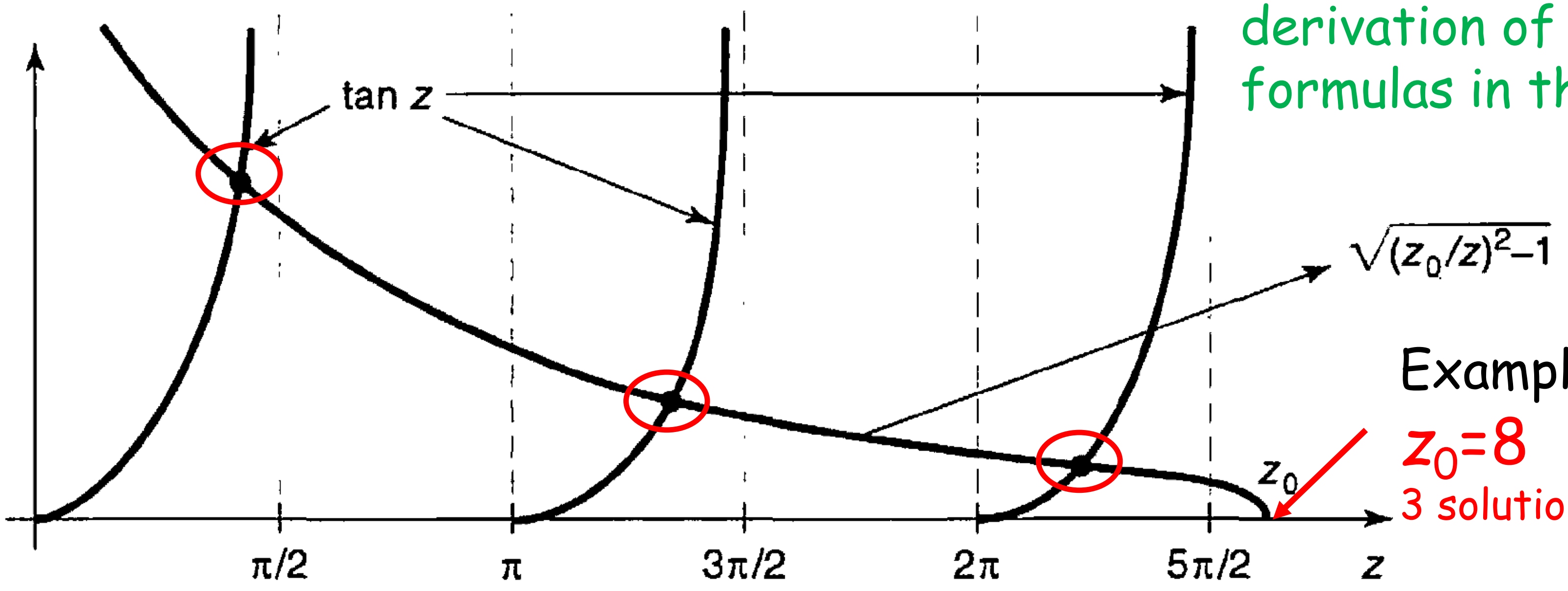
Also in general there is no need to use all the numerical values of masses, Planck constant, etc. Use clever variables.

$$z \equiv la \qquad z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0} \quad \rightarrow \quad \kappa a = \sqrt{z_0^2 - z^2}$$

Dimensionless combo

$$\rightarrow \quad \tan z = \sqrt{(z_0/z)^2 - 1}$$

We trade *E* as unknown to *z* as unknown. The derivation of these formulas in the PDF.



Example:

$z_0=8$

3 solutions

Derivation of some formulas of previous page

In the book and lecture we arrived to

$$\kappa = l\tan(la) \quad (1)$$

We introduced two definitions

$$(2)\quad z = la \quad , \quad z_0 = \frac{a}{\hbar}\sqrt{2mV_0} \quad (3)$$

where $l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$ (4) was also introduced in book + lecture

and before we used $\kappa = \frac{\sqrt{-2mE}}{\hbar}$ (5)

From (4): $l^2\hbar^2 = 2m(E+V_0) = 2mE + 2mV_0$

From (5) this is $-\kappa^2\hbar^2$ | From (3) this is $\frac{z_0^2\hbar^2}{a^2}$

Then, $l^2\hbar^2 = -\kappa^2\hbar^2 + \frac{z_0^2\hbar^2}{a^2}$

$$l^2a^2 = -\kappa^2a^2 + z_0^2$$

z^2 from (2) → $\kappa^2a^2 = z_0^2 - z^2$

From (1)

$$l^2\tan^2(la)a^2 = z_0^2 - z^2$$

From (2)

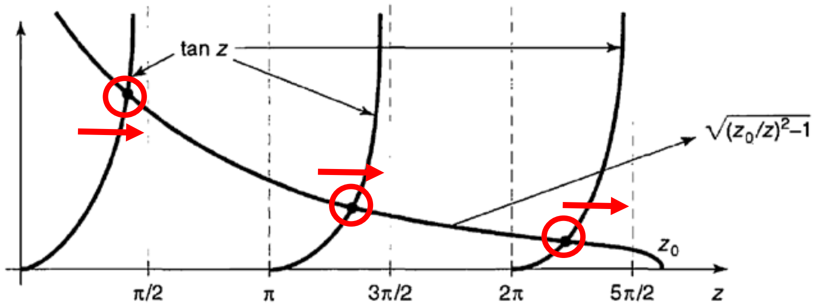
$$z^2\tan^2(z) = z_0^2 - z^2$$

$$\tan^2(z) = \frac{z_0^2}{z^2} - 1$$

$$\tan(z) = \sqrt{\frac{z_0^2}{z^2} - 1}$$

For each value of $z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$ there will be a different finite number of solutions. Consider two special limits:

(1) V_0 large i.e. a **very deep well**. This means z_0 large.



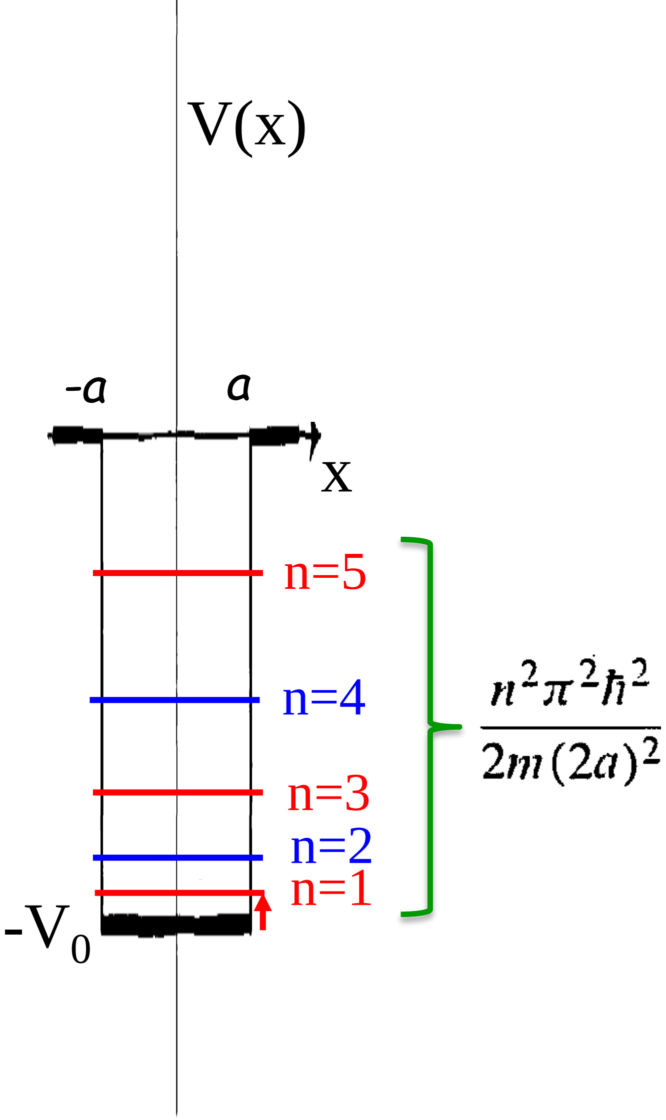
→ solutions are (n odd) $z_n = n\pi/2$

Because $z \equiv la$, then $z_n = n\pi/2$ means $l_n = n\pi/2a$, and

$$l_n \equiv \frac{\sqrt{2m(E_n + V_0)}}{\hbar}$$

thus (n odd)

$$E_n + V_0 \cong \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

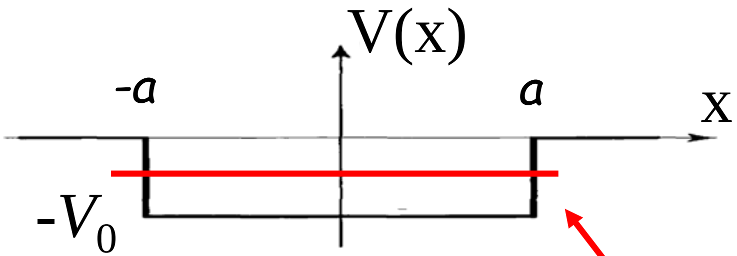


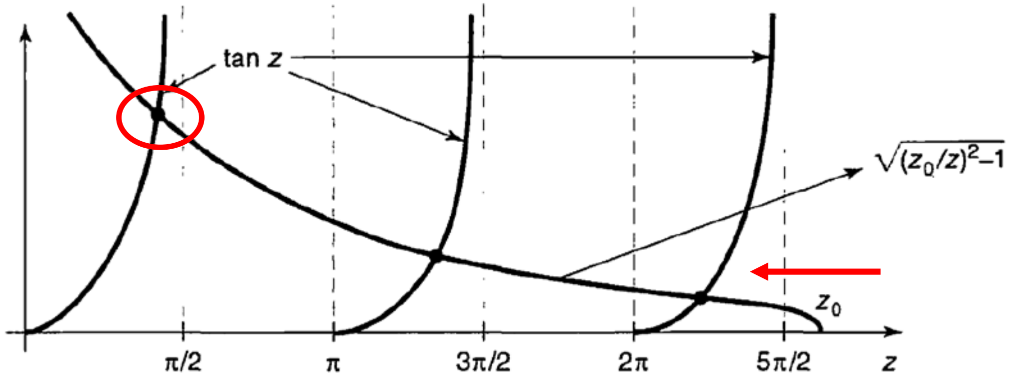
We found "half" the solutions (those "even" under x→-x, i.e. *n* odd) that we had found before for the infinite square well assuming width "2*a*".

For any large but finite V_0 the number of solutions will be large but finite as well.

The red n=1,3,5,… is what we found today.
The blue n=2,4,… you will do in HW.

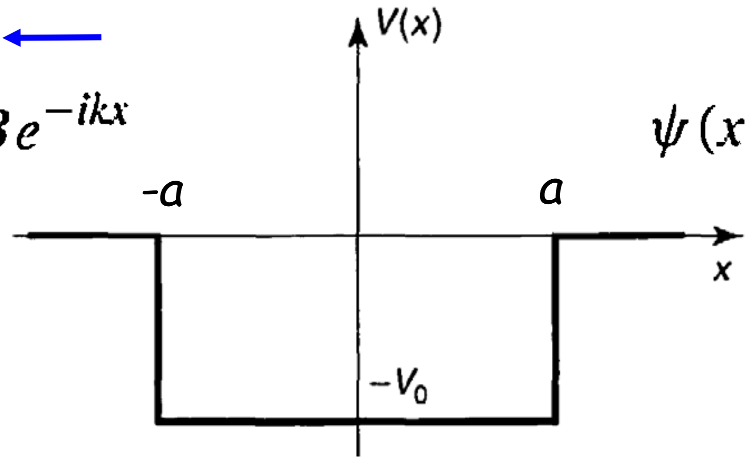
(2) V_0 small i.e. a **very shallow well**. This means z_0 small.





As z_0 is reduced, the number of solutions decreases but always one survives no matter how weak the potential is!

Consider now the scattering states $E>0$ (unrestricted).



left: $\psi(x) = Ae^{ikx} + Be^{-ikx}$, $\quad k \equiv \frac{\sqrt{2mE}}{\hbar}$

right: $\psi(x) = Fe^{ikx}$

inside: $\psi(x) = C\sin(lx) + D\cos(lx)$, $\quad l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$

Continuity of ψ and $d\psi/dx$ at $x=-a$:

$$\begin{cases} Ae^{-ika} + Be^{ika} = -C\sin(la) + D\cos(la) \\ ik[Ae^{-ika} - Be^{ika}] = l[C\cos(la) + D\sin(la)] \end{cases}$$

Continuity of ψ and $d\psi/dx$ at $x=+a$:

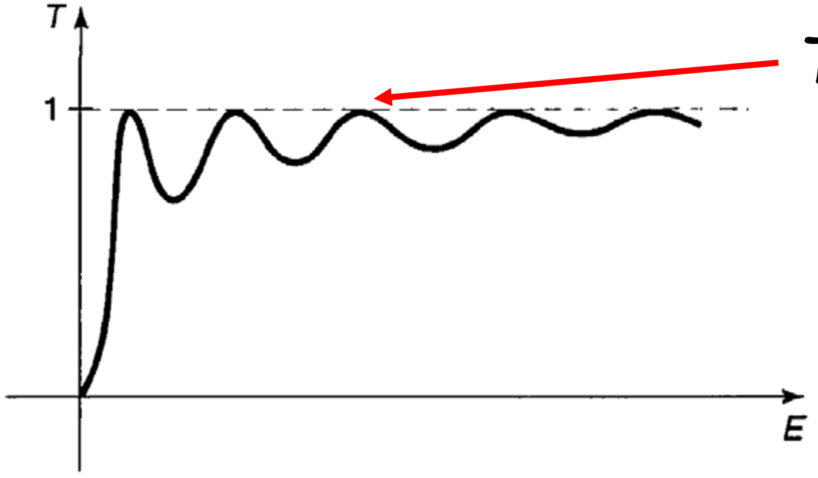
$$\begin{cases} C\sin(la) + D\cos(la) = Fe^{ika} \\ l[C\cos(la) - D\sin(la)] = ikFe^{ika} \end{cases}$$

This system of 4 equations and 5 unknowns can be solved as for the delta potential via ratios.

Recall $T = |F|^2/|A|^2$ is the transmission coefficient

The result is

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$



T=1 if $\frac{2a}{\hbar}\sqrt{2m(E_n+V_0)} = n\pi$

Thus, for some particular energies there is perfect transmission !! And they happen to be the solutions of the infinite square well. Weird QM!

$$E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

There are ghostly "remnants" of the infinite well …

SOME MOVIES SHOWN TO STUDENTS …

Movie 1: probability shown, intermediate weird state with oscillations, plotting prob and potential together is a bit confusing, not same units.
http://sces.phys.utk.edu/~dagotto/QuantumMechanics/Animation/movie_1.mp4
From original web site: "*FDTD simulation of a Gaussian wave packet with kinetic energy of 500 eV. The potential barrier has a height of 600 eV, and a thickness of 25 pm. The wave packet partially tunnels through the barrier, giving a total probability of about 17% of finding the particle on the other side.*"

Movie 2: Re and Im out of phase causing smooth probability, until collision occurs. Note vgroup and vphase are different.
http://sces.phys.utk.edu/~dagotto/QuantumMechanics/Animation/movie_2.mp4
From original web site: *"FDTD simulation of a Gaussian wave packet with kinetic energy of 500 eV. The potential barrier has a height of 600 eV, and a thickness of 25 pm. The black line represents the real part of the wave function, while the red line represents the imaginary part. The actual probability amplitude is found by taking the magnitude-square of the total wave function. The wave packet partially tunnels through the barrier, giving a total probability of about 17% of finding the particle on the other side. This simulation was written and rendered in Matlab".*