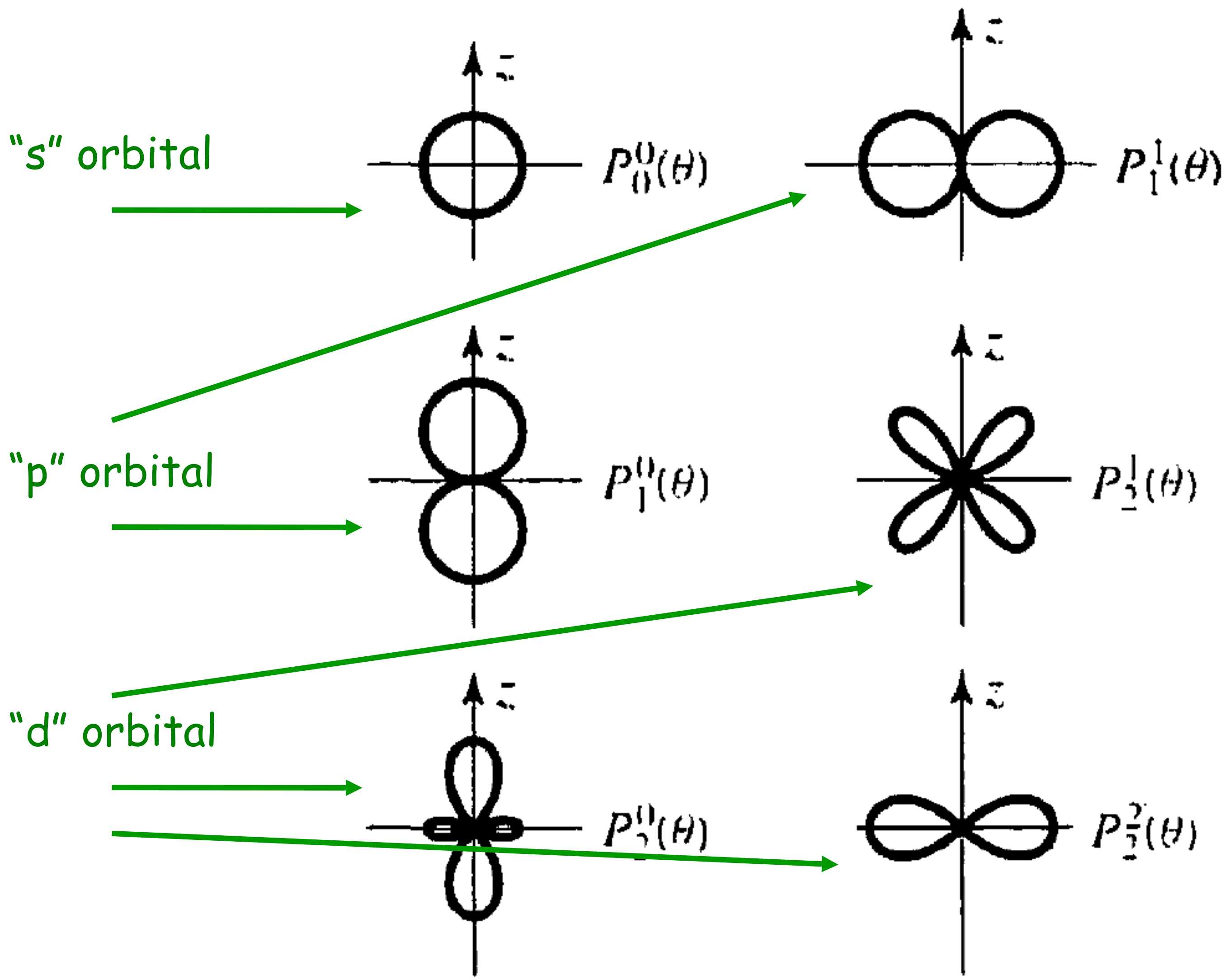
If you graph these functions, the "orbitals" that you have seen many times before since high school start appearing!



The final "touch" requires putting all together

$$\boxed{Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)}$$

where $\boxed{\Theta(\theta) = A P_l^m(\cos\theta)}$ and $\boxed{\Phi(\phi) = e^{im\phi}}$

with the "quantization" condition:

$$\boxed{l = 0,\ 1,\ 2, \ldots; \quad m = -l,\ -l+1,\ \ldots,\ -1,\ 0,\ 1,\ \ldots,\ l-1,\ l}$$

We also need to normalize using $d^3\mathbf{r} = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\int |\psi|^2 r^2 \sin\theta\, dr\, d\theta\, d\phi = \underbrace{\int |R|^2 r^2\, dr}_{=1} \underbrace{\int |Y|^2 \sin\theta\, d\theta\, d\phi}_{=1} = 1$$

For the angular component this means:

$$\int_0^{2\pi}\int_0^{\pi} |Y|^2 \sin\theta \, d\theta \, d\phi = 1$$

By this procedure the famous **spherical harmonics** arise:

one "s"

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$$

pz

three "p"

px, py

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$$

five "d"

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$$

$$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi}$$

$$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$$

seven "f"

$$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta)$$

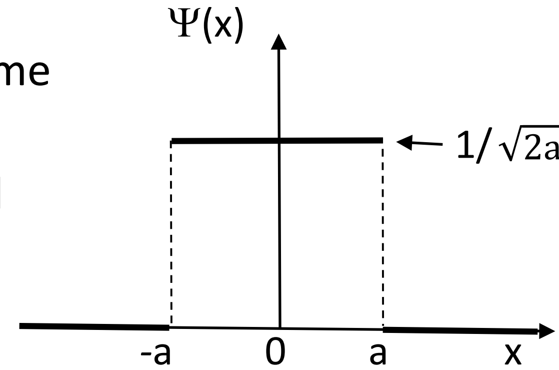
$$Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta\,(5\cos^2\theta - 1)e^{\pm i\phi}$$

$$Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta e^{\pm 2i\phi}$$

$$Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm 3i\phi}$$

Question

Consider the normalized wave function in the figure. Assume it is the wave function $\Psi(x,0)$ at time t=0 of a wave packet problem. (a) Find $\phi(k)$. Do the simple integral. (b) Now find $\Psi(x,t)$. Leave this result just expressed as an integral in k.



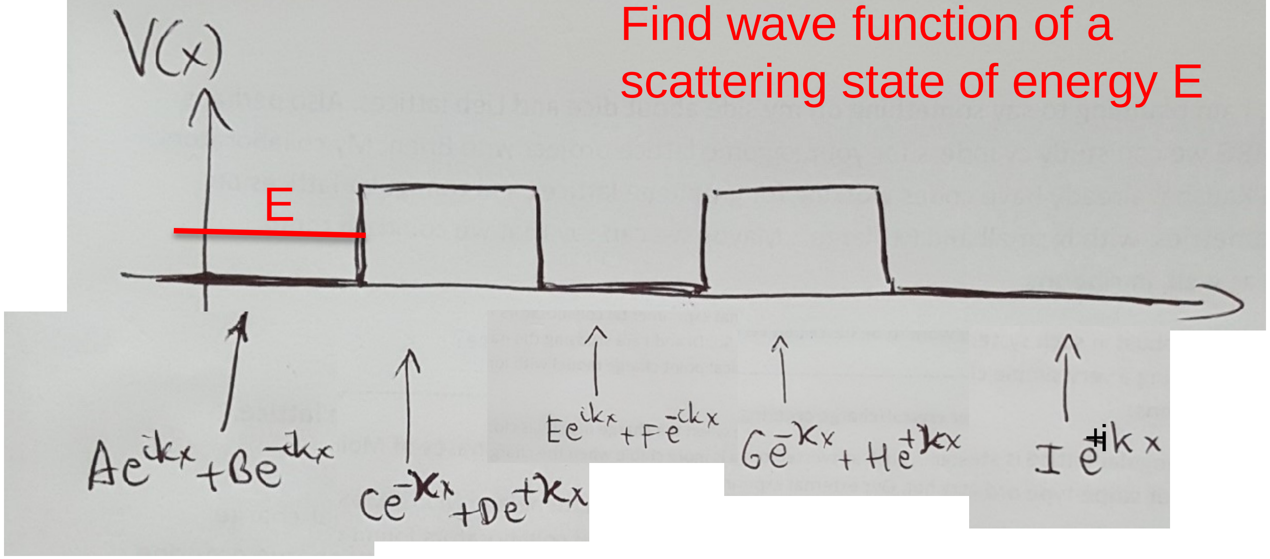
(a) $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x,0)\, e^{-ikx}\, dx = \boxed{\frac{\sin(ka)}{\sqrt{\pi a}\; k}}$$

(b) $$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk =$$

$$= \boxed{\frac{1}{\pi\sqrt{2a}} \int_{-\infty}^{+\infty} \frac{\sin(ka)}{k}\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk}$$

Question

Find wave function of a scattering state of energy E



How many unknowns? 9 A, B, C, D, E, F, G, H, I

E being scattering state is arbitrary

How many boundary condition equations?

4 frontiers, i.e. 8 (ψ and $\frac{d\psi}{dx}$ continuous)

But dividing by A we get 8 unknowns.

Question

Consider the infinite square well of width "a".

Is the wave function $\Psi(x) = \cos(\pi x/a)$ an eigenstate of the momentum operator?

Apply p operator over $\Psi(x)$ and see if you obtain a number times $\Psi(x)$

$$(-i\hbar\, d/dx)\Psi(x) = (-i\hbar)\, d/dx \cos(\pi x/a) = (i\pi\hbar/a)\sin(\pi x/a)$$

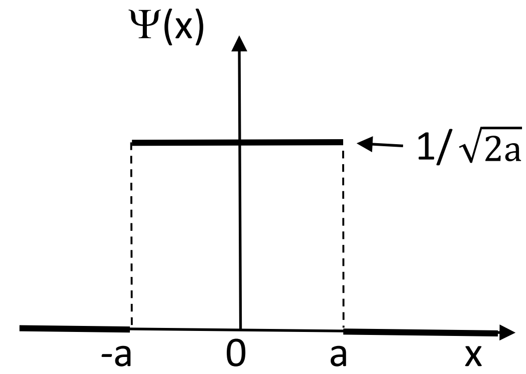
Answer: No, it is not eigenstate.

Note: you can also use Euler formulas, $(-i\hbar\, d/dx)\Psi(x) = (-i\hbar/2)\, d/dx\, (e^{i\pi x/a} + e^{-i\pi x/a}) = (-i\hbar/2)(i\pi/a)[e^{i\pi x/a} - e^{-i\pi x/a}] = (i\pi\hbar/a)\sin(\pi x/a)$

Question

Consider the normalized wave function in the figure:

(a) Find the coefficient $c(p) = \langle f_p | \Psi \rangle$, where f_p is the eigenfunction of the linear momentum with eigenvalue p.
(b) Find the probability that in a measurement, the momentum p is between 0 and ∞. Use $\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{2}$



(a) Find $c(p)$:

$$c(p) = \langle f_p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-a}^{a} e^{-i\frac{px}{\hbar}} \frac{1}{\sqrt{2a}}\,dx = \boxed{\sqrt{\frac{\hbar}{\pi a}}\,\frac{1}{p}\sin\left(\frac{pa}{\hbar}\right)}$$

(b)

$$P_{rob.\,[0,\infty]} = \int_0^\infty |c(p)|^2\,dp = \boxed{\frac{\hbar}{\pi a}\int_0^\infty \frac{1}{p^2}\sin^2\left(\frac{pa}{\hbar}\right)dp} = \boxed{\frac{1}{2}}$$

Question

$\hat{A}$ and $\hat{B}$ are Hermitian.

Thus, $\hat{A} = \hat{A}^+$ and $\hat{B} = \hat{B}^+$.

Find the Hermitian of the product operator $\hat{O} = \hat{A}\hat{B}$.

We are asked to find the operator $\hat{O}$ that satisfies $\langle O^+ \psi | \psi \rangle = \langle \psi | \hat{O} \psi \rangle$

Let us start. We need to move $\hat{A}$ and $\hat{B}$ to the other side.

$$\langle \psi | \underbrace{\hat{A}\hat{B}}_{\hat{O}} \psi \rangle \underset{\text{use } \hat{A}^+ = \hat{A}}{=} \langle \hat{A}\psi | \hat{B}\psi \rangle = \langle \underbrace{\hat{B}\hat{A}}_{\hat{O}^+} \psi | \psi \rangle$$

Then, $\boxed{\hat{O}^+ = \hat{B}\hat{A}} \neq \hat{O} = \hat{A}\hat{B}$

Question

Find $\left(\frac{d}{dx}\right)^\dagger$

$$\langle f | \frac{d}{dx} g \rangle = \int_{-\infty}^{+\infty} f^*(x) \frac{d}{dx} g(x)\, dx =$$

$$\underset{\text{by parts}}{=} \underbrace{f^*(x) g(x) \Big|_{-\infty}^{+\infty}} - \int_{-\infty}^{+\infty} \left[\frac{d}{dx} f^*(x)\right] g(x)\, dx = \int_{-\infty}^{+\infty} \left[-\frac{d}{dx} f^*(x)\right] g(x)\, dx$$

$= 0$ because we use square integrable functions
$f(x) \to 0$, $x \to \pm\infty$
$g(x) \to 0$, $x \to \pm\infty$

$$= \langle -\frac{d}{dx} f | g \rangle$$

Then $\boxed{\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}}$