Properties of inner products:

$$\langle f|g\rangle^* = \left[\int_a^b f(x)^* g(x)\,dx\right]^* = \int_a^b g(x)^* f(x)\,dx = \langle g|f\rangle$$

Then: $\langle g|f\rangle = \langle f|g\rangle^*$

Professor's suggestion: If in doubt, work with the explicit integrals as definition of inner product, instead of the <…|…> notation,

In particular: $\langle f|f\rangle = \int_a^b |f(x)|^2\,dx$

This is real and non negative, justifying the conjugation in $f(x)$ in the definition of $\langle f|g\rangle$.
$\langle f|f\rangle=1$ means $f(x)$ is normalized to 1.

In this notation, then "orthonormality" of functions is written as:

$$\langle f_m | f_n \rangle = \delta_{mn}$$

"Completeness": any function in the Hilbert space of the problem can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$$|f\rangle = \Sigma_n c_n |f_n\rangle$$

$$\langle f_m|f\rangle = \Sigma_n c_n \underbrace{\langle f_m |f_n\rangle}_{\delta_{mn}}$$

Then, the coefficients are $c_n = \langle f_n | f \rangle$ which is what we knew already from Ch. 2 i.e.

$$c_n \equiv \int_a^b f_n(x)^* f(x)\, dx$$

3.2 Observables and Hermitian Operators

Expectation values of operators, such as $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{H}\rangle$, … can be expressed in general as:

$$\langle\hat{Q}\rangle = \int \Psi^* \hat{Q} \Psi \, dx = \langle\Psi|\hat{Q}\Psi\rangle = \langle\Psi|\hat{Q}|\Psi\rangle$$

where $\hat{Q}$ is the $\hat{x}$, $\hat{p}$, $\hat{H}$, … operator.

But if $\hat{Q}$ is related with a measurement, physically we expect to obtain real numbers. Thus, for the operators of relevance we expect:

$$\langle\hat{Q}\rangle = \langle\hat{Q}\rangle^*$$

Because $\langle \hat{Q} \rangle = \langle \hat{Q} \rangle^*$ means <- from physics

$$\langle \Psi | \hat{Q}\Psi \rangle = \langle \Psi | \hat{Q}\Psi \rangle^*$$

and because $\langle f | g \rangle^* = \langle g | f \rangle$ <- from math

($g=\hat{Q}\Psi$, $f=\Psi$)

then $$\boxed{\langle \Psi | \hat{Q}\Psi \rangle = \langle \hat{Q}\Psi | \Psi \rangle}$$

Operators $\hat{Q}$ that satisfy this framed property are called **Hermitian operators**.

Thus, observables are represented by Hermitian operators.

Easy example: coordinate $\hat{x}$ operator

$$\langle \hat{x} \rangle = \underbrace{\int \Psi^* \hat{x} \Psi \, dx}_{\langle \Psi | \hat{x} \Psi \rangle} = \int (x\Psi)^* \Psi \, dx = \langle \hat{x} \Psi | \Psi \rangle$$

$$\boxed{\langle \Psi | \hat{Q} \Psi \rangle = \langle \hat{Q} \Psi | \Psi \rangle}$$

Thus, $\hat{x}$ is Hermitian

$(\hat{x})^2$ is also Hermitian, and in general all potentials $V(\hat{x})$, like harmonic oscillator, are Hermitian.

Less trivial example: momentum $\hat{p}$ operator

$$\langle \hat{p} \rangle = \underbrace{\int \Psi^* \hat{p} \Psi \, dx}_{\langle \Psi | \hat{p} \Psi \rangle} = -i\hbar \int \Psi^* \frac{d}{dx} \Psi \, dx$$

Just using definition of operator *p*

By parts

$$\int_{-\infty}^{\infty} \Psi^* \frac{d}{dx} \Psi \, dx = \underbrace{\Psi^* \Psi \Big|_{-\infty}^{\infty}}_{=0 \text{ in the Hilbert space}} - \int_{-\infty}^{\infty} \left(\frac{d}{dx}\Psi^*\right) \Psi \, dx$$

=0 in the Hilbert space

Thus, so far $\langle \Psi | \hat{p} \Psi \rangle = -i\hbar \int \Psi^* \frac{d}{dx} \Psi \, dx$

Check for Test2!

$$= (-i\hbar)\left(-\int_{-\infty}^{\infty} \left(\frac{d}{dx}\Psi^*\right) \Psi \, dx\right) = \int_{-\infty}^{\infty} \left(-i\hbar \frac{d}{dx} \Psi\right)^* \Psi \, dx = \langle \hat{p}\Psi | \Psi \rangle$$

Because (-i)(-1)=(-i)*. Or (ab)*=a*b*

Because $\langle \Psi | \hat{p} \Psi \rangle = \langle \hat{p} \Psi | \Psi \rangle$

then, $\hat{p}$ is a "Hermitian operator"

$-i \frac{d}{dx}$ is Hermitian ($\hbar$ is just a constant)

$-\frac{d}{dx}$, then, is not Hermitian alone

$(-i \frac{d}{dx})^2$ is Hermitian

(product of two Hermitians is Hermitian)

$\hat{H} = \hat{p}^2/2m + V(\hat{x})$ is Hermitian