For the definition of Hermitian operators most books require using two different functions f(x) and g(x). Steps are the same as we did, no worries. **This more general form is sometimes useful, as the practice exam will show.**

$$\langle f|\hat{Q}g\rangle = \langle\hat{Q}f|g\rangle$$

If $\hat{Q}$ is NOT Hermitian, like d/dx, then the **definition** of Hermitian of an operator (a.k.a. Hermitian conjugate, or adjoint) is the operator $\hat{Q}^{\dagger}$ that satisfies:

$$\langle f|\hat{Q}g\rangle = \langle\hat{Q}^{\dagger}f|g\rangle$$

Examples: $(d/dx)^{\dagger} = -(d/dx)$, $(i)^{\dagger} = -i$ (for a complex number "dagger" is the same as "conjugation"), $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$ see next.

PROBLEM 2, practice Test2, 2022: (a) Consider two arbitrary operators $\hat{A}$ and $\hat{B}$ in a Hilbert space. Consider now their product $\hat{A}\hat{B}$. Find the Hermitian of the product i.e. find $(\hat{A}\hat{B})^\dagger$, using twice the straightforward definition of Hermitian of an operator, discussed in lecture Oct. 26 i.e. $\langle f|\hat{Q}g\rangle = \langle\hat{Q}^\dagger f|g\rangle$

$$\langle f|\hat{A}\hat{B}g\rangle = \langle\hat{A}^+ f|\hat{B}g\rangle = \langle\hat{B}^+\hat{A}^+ f|g\rangle$$

Thus, based on $\langle f|\hat{Q}g\rangle = \langle\hat{Q}^+ f|g\rangle$ we conclude $(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$

(b) Consider the operator $\hat{O}=i\hat{p}$, where i is the imaginary unit and $\hat{p}$ the standard momentum operator which you already know is Hermitian. Is the operator $\hat{O}$ Hermitian in a Hilbert space? Solve this problem by any procedure you wish. But you cannot answer tersely just “yes” or “no”: need to justify in some logical manner your answer.

$$(\hat{i}\,\hat{p})^\dagger = \hat{p}^\dagger\,\hat{i}^\dagger = \hat{p}\,(-i) = -i\hat{p}$$

No, the operator is not Hermitian due to the extra sign

Alternative solution of (b): What is the Hermitian of $i\hat{p}$ i.e. $(i\hat{p})^\dagger$?

$$i\hat{p} = +\hbar\frac{d}{dx}$$

$$\langle i\hat{p}\rangle = \underbrace{\int \Psi^* i\hat{p}\,\Psi\,dx}_{\langle\Psi|i\hat{p}\,\Psi\rangle} = +\hbar\int \Psi^*\frac{d}{dx}\Psi\,dx$$

Just using definition of operator *ip* and using i$\times$(-i)=+1.

By parts

$$\int_{-\infty}^{\infty}\Psi^*\frac{d}{dx}\Psi\,dx = \underbrace{\Psi^*\Psi\Big|_{-\infty}^{\infty}}_{=0\text{ in the Hilbert space}} - \int_{-\infty}^{\infty}\left(\frac{d}{dx}\Psi^*\right)\Psi\,dx$$

This framed green formula is the same as used before to show the momentum is Hermitian.

Thus, so far $\langle\Psi|i\hat{p}\,\Psi\rangle = +\hbar\int\Psi^*\frac{d}{dx}\Psi\,dx$

$$=(+\hbar)\left(-\int_{-\infty}^{\infty}\left(\frac{d}{dx}\Psi^*\right)\Psi\,dx\right) = \int_{-\infty}^{\infty}\left(-\hbar\frac{d}{dx}\Psi\right)^*\Psi\,dx = \langle i\hat{p}\Psi|\Psi\rangle$$

Then: $(i\hat{p})^\dagger = -i\hat{p}$ i.e. not Hermitian, but anti-Hermitian.

3.2.2 "Determinate" States

The "stationary states" of the $\hat{H}$ Hamiltonian, Ψ_n, have a **sharp** energy E_n. Can we do the same for other operators $\hat{Q}$?

Similarly as when we used to write $\hat{H}\Psi_n(x)=E_n\Psi_n(x)$, we define *eigenfunctions of hermitian* $\hat{Q}$, $f_q(x)$, such that

$$\hat{Q}\, f_q(x) = q\, f_q(x) \; [q=q_1,q_2,q_3, \ldots]$$

"q" is just a number, called the *eigenvalue* of the operator $\hat{Q}$. The reason for the language is the similarity to a matrix operation $\mathbf{Ta} = \lambda\mathbf{a}$ (see Appendix A.5). There are many $\lambda = \lambda_1, \lambda_2, \ldots$

Example: in Chapter 4 we will discuss eigenfunctions of the angular momentum operator $\hat{L}$, the "spin" $\hat{S}$, etc.

Example: the momentum operator $\hat{p}$.

First we need an eigenfunction $f_p(x)$ such that

$$\frac{\hbar}{i}\frac{d}{dx} f_p(x) = p f_p(x)$$

($\hat{p}$ operator: $\frac{\hbar}{i}\frac{d}{dx}$; eigenfunction: $f_p(x)$; eigenvalue: p)

Check for Test2!

Solution is very easy (but normalization is unusual because they are not normalizable -> page 100):

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$\langle f_p(x) | f_{p'}(x) \rangle = \delta(p-p')$

Given an arbitrary wave function $\Psi(x,t)$, if we would time we would discuss the analog of $c_n = \langle f_n | f \rangle$

$$c(p) = \langle f_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

($f_p^*(x)$)

For continuous eigenvalues, $c(p)$ no longer "a number from 0 to 1".

PROBLEM 4, practice Test2, 2022: Consider the wave function

$\psi(x) = \sin(\pi x/a) - i\cos(\pi x/a)$

(a) Prove that $\psi(x)$ is an **eigenfunction** of the linear momentum operator.

(b) Provide the associated eigenvalue.

(a) The operator is $-i\hbar \frac{d}{dx} = \hat{p}$, then

$$\hat{p}\psi = (-i\hbar)\frac{d}{dx}\left[\sin\left(\frac{\pi x}{a}\right) - i\cos\left(\frac{\pi x}{a}\right)\right] =$$

$$= (-i\hbar)\left[\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right) - i\frac{\pi}{a}\left(-\sin\left(\frac{\pi x}{a}\right)\right)\right] = \left(\frac{-i\hbar\pi}{a}\right) \times$$

$$\times\left[\cos\left(\frac{\pi x}{a}\right) + i\sin\left(\frac{\pi x}{a}\right)\right] = \left(\frac{\hbar\pi}{a}\right)\left[\sin\left(\frac{\pi x}{a}\right) - i\cos\left(\frac{\pi x}{a}\right)\right]$$

Yes, it is eigenfunction.

(b) Eigenvalue is $\left(\frac{\hbar\pi}{a}\right)$.

There are many theorems for Hermitian operators, the operators that matter for observables, that I will NOT prove:

(1) The eigenvalues q are real, like the energies E_n were (see page 98 for a 100% math proof). If you measure $\hat{Q}$ in any $\Psi(x,t)$, you will get one of the q's.

(2) The eigenfunctions $f_q(x)$ for different q's are orthonormal -- sometimes using Kronecker delta, sometimes Dirac delta -- like $\Psi_n(x)$ for energies were.

(3) The eigenfunctions are complete, like $\Psi_n(x)$ for energies were.

Caveat: careful with degenerate states i.e. those with the same eigenvalue q.

Generalized uncertainty principle. Proof in pages at the end of Ch3 if you are interested in.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Assumes <..> is in a normalized to 1 state, and both operators must be Hermitian.

Example, if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, then $[\hat{x}, \hat{p}] = i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2 = \left(\frac{\hbar}{2} \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

It can also be trivial: if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{x}^2$, then $\sigma_x \sigma_{x2} = 0$ because they conmute