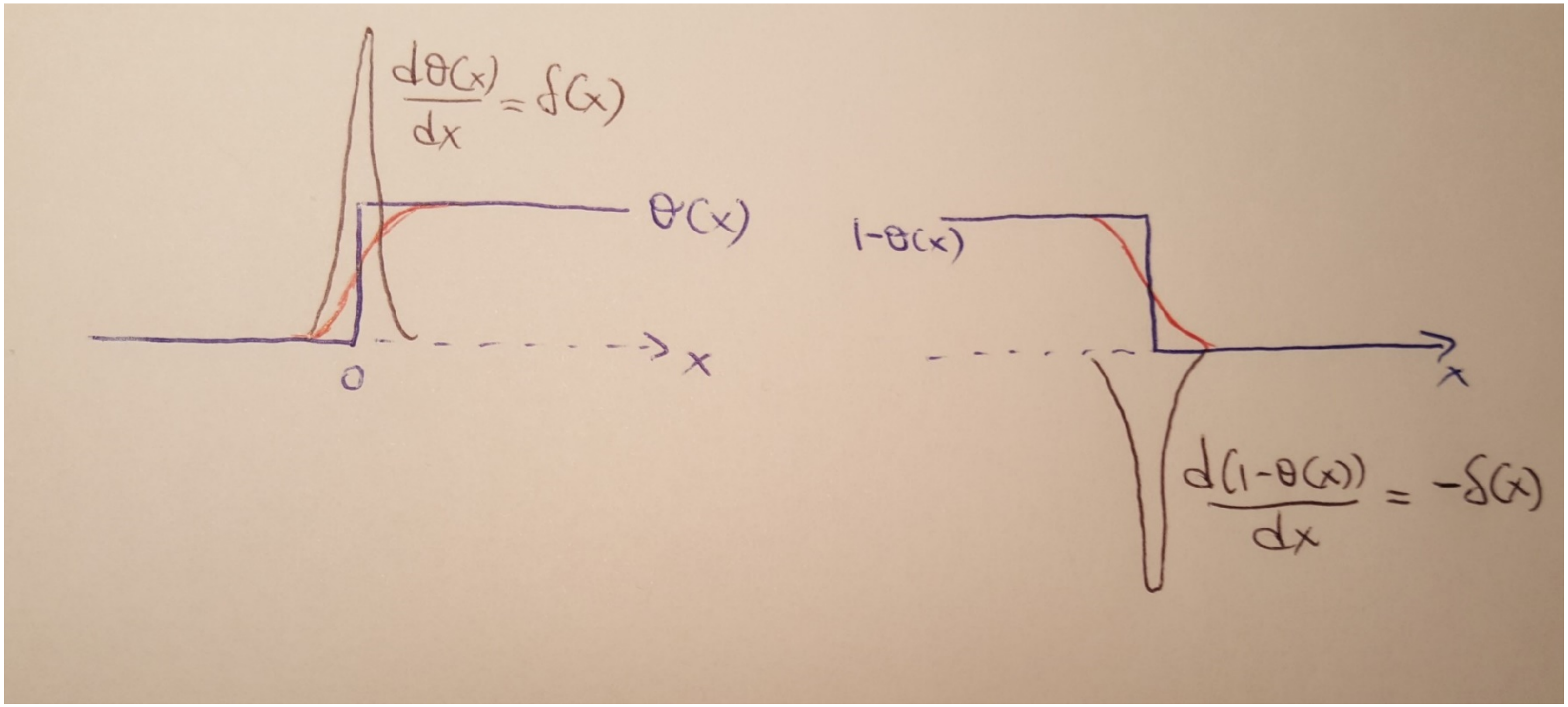
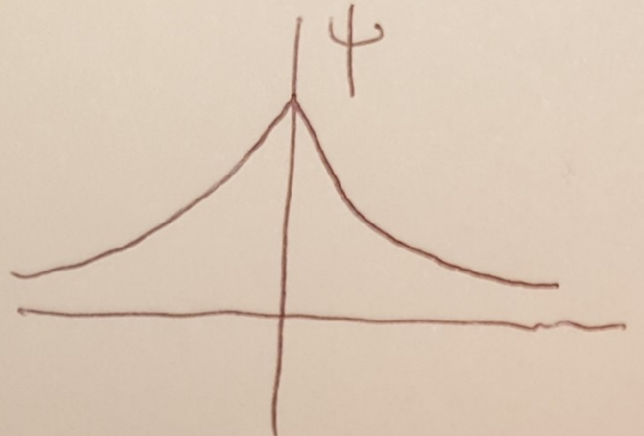
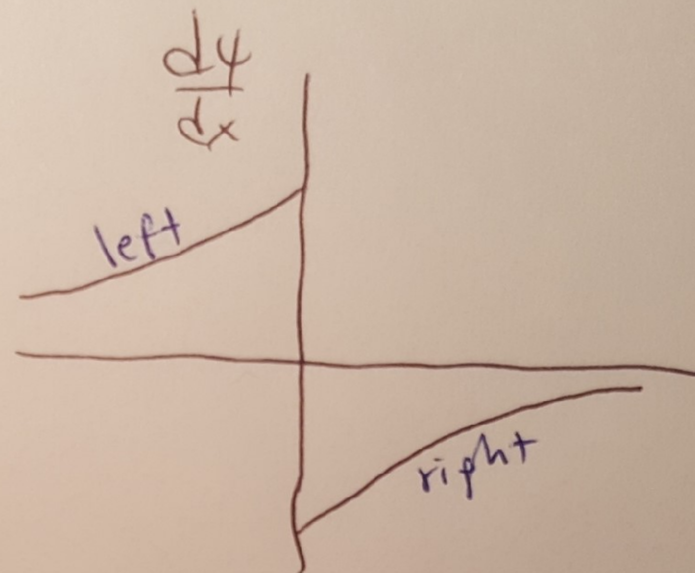
About HW5 2.24:







$$\frac{d\psi}{dx} = \theta(x)\underbrace{\frac{d\psi}{dx}}_{\text{right}} + [1-\theta(x)]\underbrace{\frac{d\psi}{dx}}_{\text{left}}$$

and then take derivatives using

$$\frac{d\theta(x)}{dx} = \delta(x) \, , \quad \frac{d(1-\theta(x))}{dx} = -\delta(x)$$

Chapter 3: Formalism

Consider a 3D vector using real numbers:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\mathbf{a} = \sum_{n=1}^{3} a_n \, \mathbf{e}_n$$

The "dot product" is defined as:

$$\mathbf{a.b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

It is not difficult to imagine a generalization to N dimensions and to complex numbers:

$$|\alpha\rangle \rightarrow \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

$$\mathbf{a} = \sum_{n=1}^{N} c_n \, \mathbf{e}_n$$

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_N^* b_N$$

Linear transformations, represented by matrices, act on vectors. For instance, a rotation.

$$|\beta\rangle = T|\alpha\rangle \rightarrow \mathbf{b} = \mathbf{Ta} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & \cdots & t_{2N} \\ \vdots & \vdots & & \vdots \\ t_{N1} & t_{N2} & \cdots & t_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

Like an "operator" acting on a vector **a**

In Quantum Mechanics, wave functions, expressed as a sum over stationary states, are like an N → infinite dimensional vector

$$\Psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

Similar to a full vector **a**

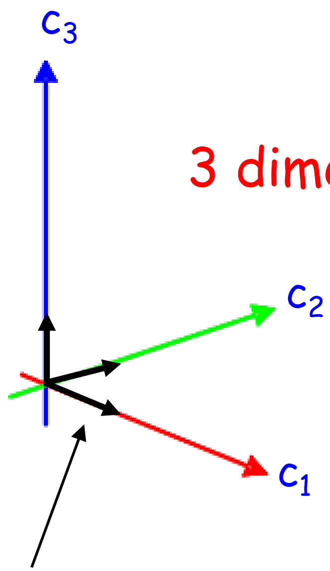
Like unit vectors $\mathbf{e}_n$

Of the set of possible functions, the subset that can be normalized ("square-integrable functions") is called a Hilbert space.

From many lectures back …

Cartesian axes

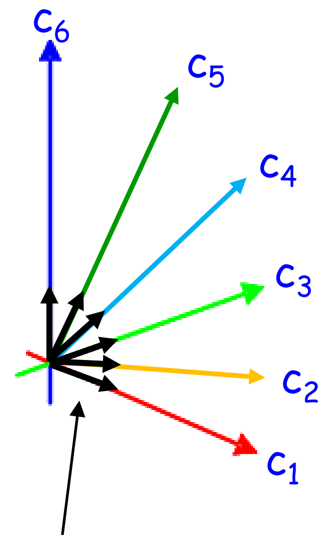
$$r = \sum_{n=1}^{3} c_n \, e_n$$



3 dimensions

Unit vectors e_1, e_2, e_3.

Any vector can be expanded in the orthonormal basis e_1, e_2, e_3.



$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

Square well solutions

∞ dimensions

"Unit vectors" are $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \ldots$

Any wave function can be expanded in the orthonormal basis ψ_n

All these properties are not pathological of the square well or the harmonic oscillator but generic: all have Hilbert Spaces.

For example, in the infinite square well of width "a" the set of functions that are normalizable i.e.

$$\int_0^a |f(x)|^2 \, dx < \infty,$$

define the "Hilbert space of the infinite square well". In general, in the [a,b] interval (which could be [-∞,+∞]):

$$\int_a^b |f(x)|^2 \, dx < \infty$$

We say the function is "square integrable"

$$\langle f | g \rangle \equiv \int_a^b f(x)^* g(x) \, dx$$

This is the "inner product" of f(x) with g(x) (analogous of the dot product in 3D)

Properties of inner products:

$$\langle f|g\rangle^* = \left[\int_a^b f(x)^* g(x)\, dx\right]^* = \int_a^b g(x)^* f(x)\, dx = \langle g|f\rangle$$

Then: $$\langle g|f\rangle = \langle f|g\rangle^*$$

Prof's suggestion: If in doubt, work with the explicit integrals as definition of inner product, instead of the <…|…> notation,

In particular: $\langle f|f\rangle = \int_a^b |f(x)|^2\, dx$

It is real and non negative, justifying the conjugation in *f(x)* in the definition of <*f*|*g*>.
<*f*|*f*>=1 means normalized to 1.

In this notation, then "orthonormality" of functions is written as:

$$\langle f_m | f_n \rangle = \delta_{mn}$$

"Completeness": any function in the Hilbert space of the problem can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$$|f\rangle = \Sigma_n c_n |f_n\rangle$$

$$\langle f_m | f\rangle = \Sigma_n c_n \underbrace{\langle f_m | f_n \rangle}_{\delta_{mn}}$$

Then, the coefficients are $c_n = \langle f_n | f \rangle$ which is what we knew:

$$c_n \equiv \int_a^b f_n(x)^* f(x)\, dx$$

One last observation: the Schwarz inequality

In three dimensions, it is obvious that:

$$(\mathbf{a}.\mathbf{b})^2 = |\,\mathbf{a}\,|^2 \ |\,\mathbf{b}\,|^2 (\cos \phi)^2 \leq |\,\mathbf{a}\,|^2 \,|\,\mathbf{b}\,|^2$$

In N dimensions, it can be shown that:

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$$

or, more explicitly, and taking square roots:

$$\left|\int_a^b f(x)^* g(x)\, dx\right| \leq \sqrt{\int_a^b |f(x)|^2\, dx \int_a^b |g(x)|^2\, dx}.$$