1.3 Probability

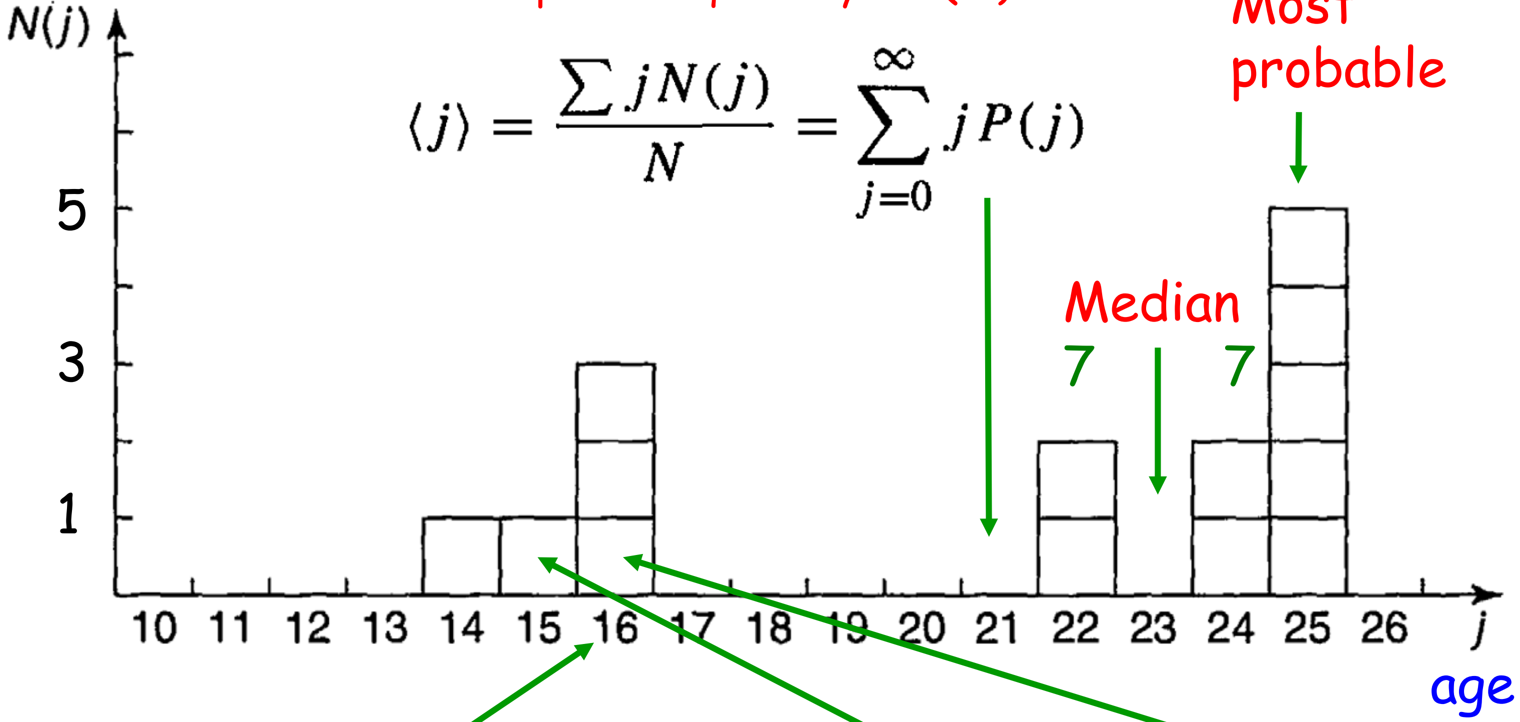
$$N = \sum_{j=0}^{\infty} N(j) = 14$$

Number of people of age j

Average or mean or "expectation value" (most important quantity in QM)

$$\langle j \rangle = \frac{\sum jN(j)}{N} = \sum_{j=0}^{\infty} jP(j)$$

Most probable



$$P(j) = \frac{N(j)}{N}$$

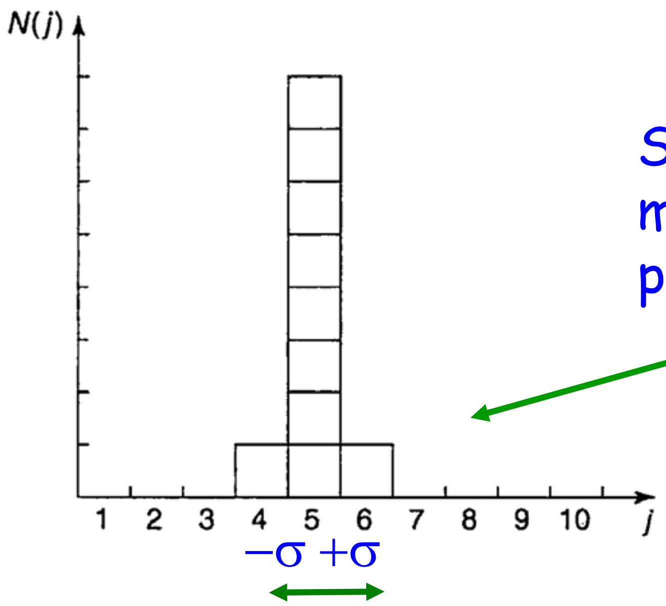
example $P(16) = 3/14$

$P(15) = 1/14$, $P(16) = 3/14$

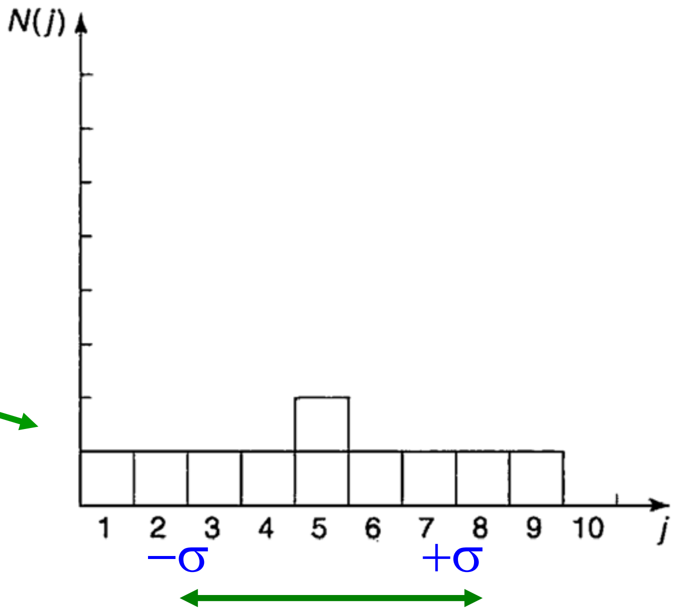
$P(15)+P(16)=4/14$

$$\sum_{j=0}^{\infty} P(j) = 1.$$

In addition to average, median, and most probable, there is another very important quantity to characterize a histogram: the standard deviation (or width). Like "error bars".



Same average, median, most probable = 5



$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

Standard deviation

$$\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2 P(j)$$

When we use continuous variables (say x instead of j) then we have to talk about a probability density.

$$\left\{\begin{array}{l}\text{probability that an individual (chosen}\\ \text{at random) lies between } x \text{ and } (x+dx)\end{array}\right\} = \rho(x)\,dx$$

$$P_{ab} = \int_a^b \rho(x)\,dx \qquad 1 = \int_{-\infty}^{+\infty} \rho(x)\,dx$$

$$\langle x\rangle = \int_{-\infty}^{+\infty} x\rho(x)\,dx \qquad \sigma^2 \equiv \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$$

So $|\Psi(x,t)|^2$ is a probability density.

1.4 Normalization

Based on the statistical interpretation of $|\psi(x,t)|^2$, its integral has to be 1 because the particle must be somewhere.

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx = 1$$

Thus, normalizing to 1 is just common sense.

If we are given a not normalized wave function $f(x,t)$, we simply choose a multiplicative constant A such that

$$|A|^2 \int_{-\infty}^{+\infty} |f(x,t)|^2\, dx = 1$$

The normalization is up to a constant phase factor that, usually, has no physical importance.

Notes: If $\psi=0$, then the integral can never be 1.
If the integral $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx$ diverges it cannot be normalized.

We will, mainly, deal with **square integrable** wave functions.

For this to make sense, once we normalize to 1 at t=0 the normalization must remain 1 at all times. Otherwise particles will be created or removed varying t. Is this true?

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2\,dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

But $\underbrace{\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi}$ and $\underbrace{\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*}$

Obtained by multiplying all terms in Sch. Eq. by $-i/\hbar$

The terms with V cancel out after addition.

Check!

Remember 1/i = -i because $i^2=-1$

Check!

Check!

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]$$

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \cancel{\int_{-\infty}^{+\infty}}\cancel{\frac{\partial}{\partial x}}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]dx$$

Integrating a derivative, cancels the two operations.

$$= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty} = 0$$

If $\psi \to 0$ as $x \to (\pm)$ infinity

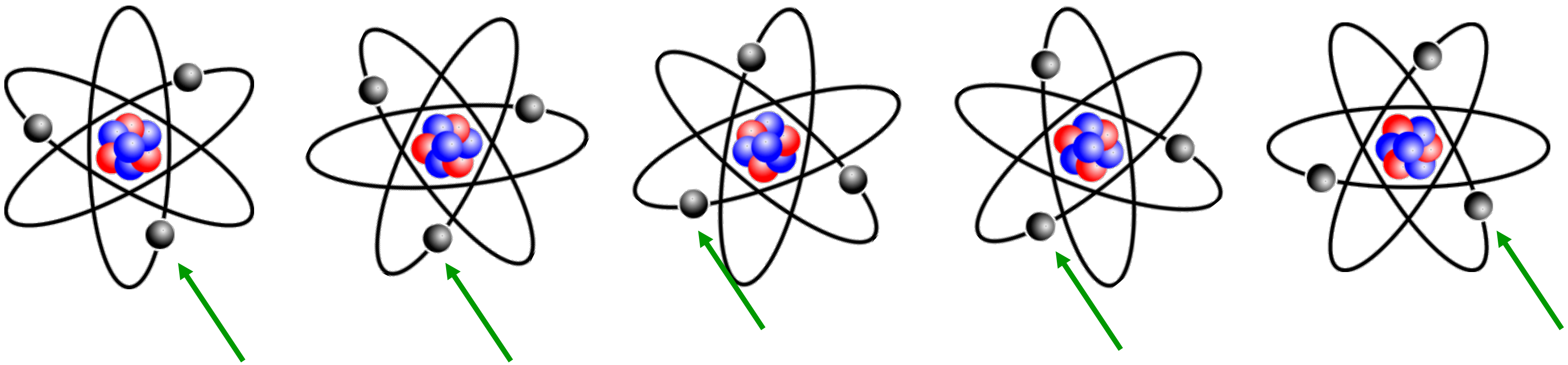
If ψ is normalized at t=0, it remains normalized at all times.
Crucial for all this to make sense!

Expectation value of x

$$\boxed{\langle x\rangle = \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2\,dx}$$

Note: $\langle x\rangle(t)$ can be time dependent.

Interpretation: $\langle x\rangle$ is the average of measurements performed on an ensemble of identical systems.



Expectation value of momentum p

<velocity>

See two pages back

$$\langle v\rangle = \frac{d\langle x\rangle}{dt} = \int x\frac{\partial}{\partial t}|\Psi|^2\,dx = \frac{i\hbar}{2m}\int \underbrace{x}_{f}\frac{\partial}{\partial x}\underbrace{\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)}_{g}dx$$

(See Chs. 2 and 3)

Integration by parts
(back cover of book):

$$\int_a^b f\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}g\,dx + fg\Big|_a^b$$

Check!

$$\frac{d\langle x\rangle}{dt} = -\overbrace{\frac{i\hbar}{2m}\int}^{df/dx=dx/dx=1}\overbrace{\left(\Psi^*\frac{\partial\Psi}{\partial x}-\underbrace{\frac{\partial\Psi^*}{\partial x}\Psi}_{\text{By parts again}}\right)}^{g}dx$$

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int \Psi^*\frac{\partial\Psi}{\partial x}\,dx$$

Check!

$$\boxed{\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx}$$

In summary, for $\langle x \rangle$ and $\langle p \rangle$ we find

$$\langle x \rangle = \int \Psi^* \underbrace{x}_{} \Psi \, dx \qquad \langle p \rangle = \int \Psi^* \underbrace{\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)}_{} \Psi \, dx$$

x "operator" is just "multiply by x

p "operator" is more complicated!

Many other operators are functions of x and p.
For instance, for the kinetic energy $T=p^2/2m$ use:

$$p^2 = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) = -\hbar^2\frac{\partial^2}{\partial x^2}$$

By this procedure a "**dictionary**" between classical and quantum quantities can be established.