$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

Or, using the other order, we get (left as exercise):

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

Theorem: if ψ satisfies $\hat{H}\psi = E\psi$, then $\hat{H}(\hat{a}_+\psi) = (E + \hbar\omega)(\hat{a}_+\psi)$

$$\hat{H}(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{1}{2}\hat{a}_+\right)\psi$$

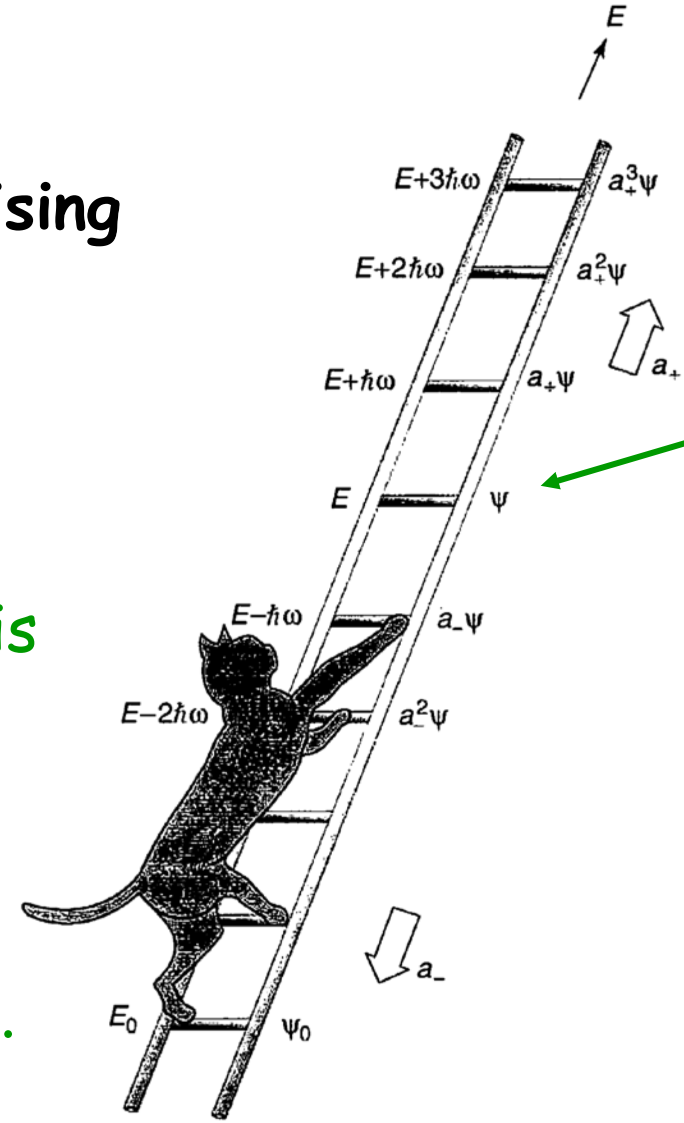
$$= \hbar\omega\hat{a}_+\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)\psi = \hat{a}_+\left[\hbar\omega\left(\hat{a}_-\hat{a}_+ + 1 - \frac{1}{2}\right)\psi\right]$$

$$= \hat{a}_+(\hat{H} + \hbar\omega)\psi = \hat{a}_+(E + \hbar\omega)\psi = (E + \hbar\omega)(\hat{a}_+\psi).$$

If I know one solution, I know another solution …

Another theorem: if ψ satisfies $\hat{H}\psi = E\psi$, then $\hat{H}(\hat{a}_-\psi) = (E - \hbar\omega)(\hat{a}_-\psi)$ *(left as exercise)*

$\hat{a}_+$ is the **raising** operator



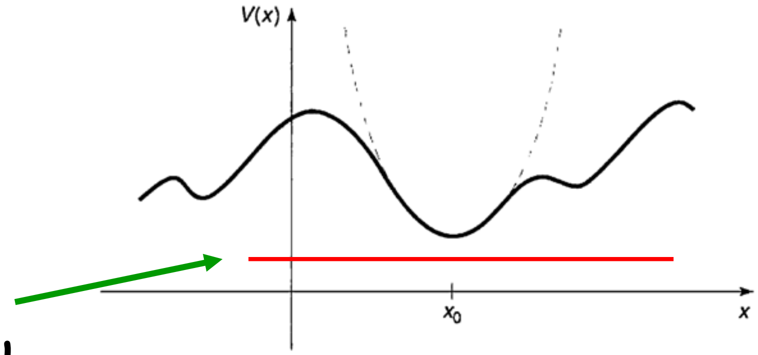
If I am given one solution, I get infinite more.

Equally spaced energies! This is why it works.

Not general: valid for oscillators only.

$\hat{a}_-$ is the **lowering** operator

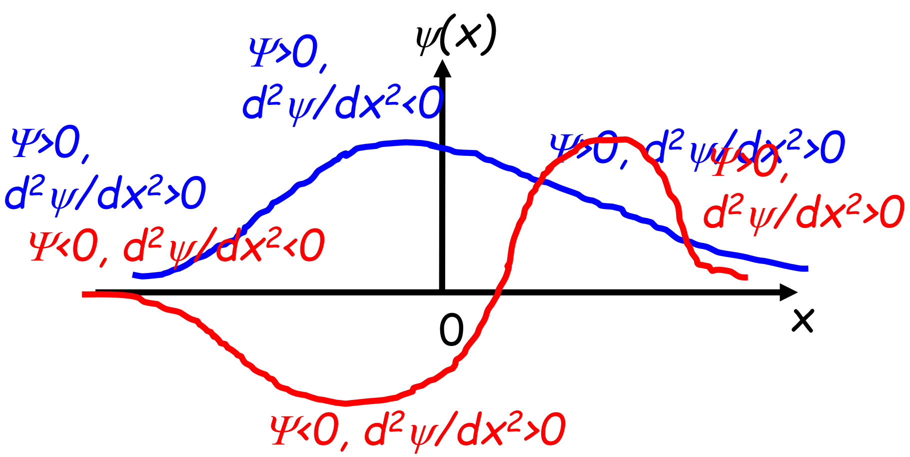
However, there is a problem:
the energy cannot continue going down!



Theorem: E less than V(x) cannot happen

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \longrightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$$

If this is >0 for all x, then ψ and $d^2\psi/dx^2$ must have same sign.



Because the energy cannot continue going down forever, the chain down must stop…

Eventually once true ground state ψ_0 is reached, then $\hat{a}_- \psi_0$ must be 0. We can use this condition to find ψ_0 .

$$\underbrace{\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx}+m\omega x\right)}_{\hat{a}_-}\psi_0 = 0 \longrightarrow \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0$$

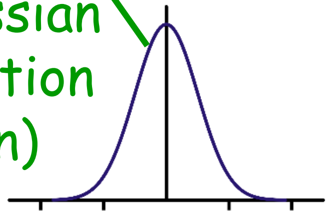
$$\int \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx \quad \Rightarrow \quad \ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + \text{constant},$$

$$\boxed{\psi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}.}$$

After normalization:
(left as exercise, use Gaussian integrals in back cover of book)

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Gaussian function (even)



What is the energy E_0?

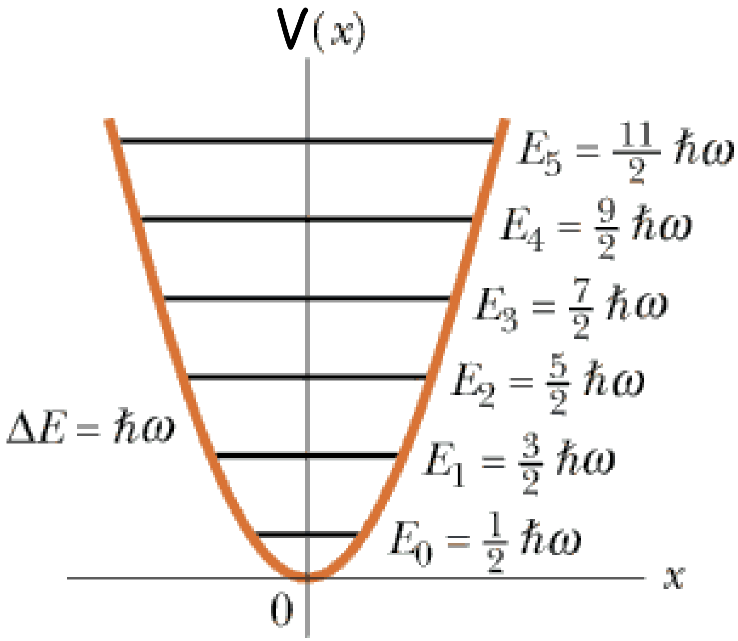
$$\hbar\omega(\hat{a}_+\hat{a}_- + 1/2)\psi_0 = E_0\psi_0$$

=0 ($\hat{a}_-\psi_0$)

$\hat{H}$ (Hamiltonian)

Then: $$E_0 = \frac{1}{2}\hbar\omega$$

>0 as expected.
"Zero point energy".
The harmonic oscillators are never still!



Equally spaced levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Solutions are

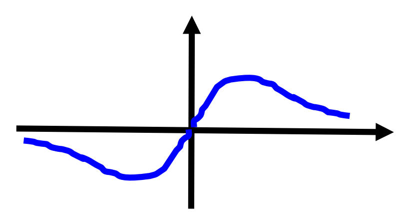
$$\psi_n(x) = A_n(\hat{a}_+)^n \psi_0(x)$$

Example 2.4: construct state 1

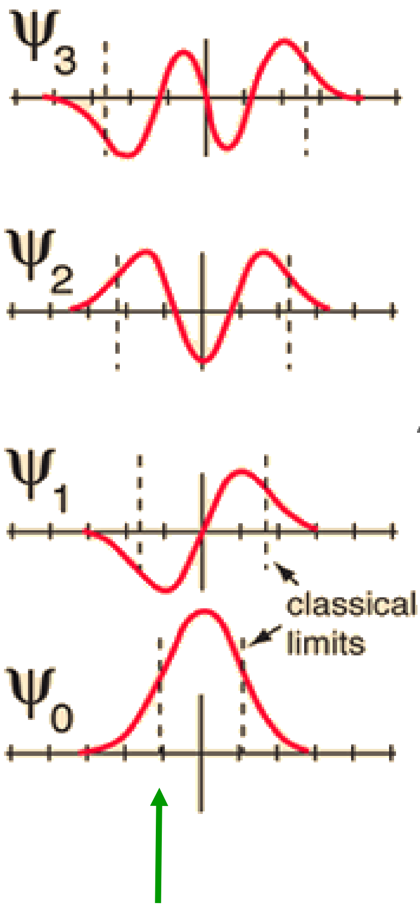
Brings down an x

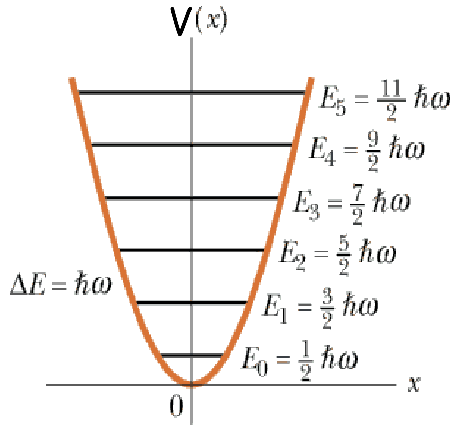
$$\psi_1(x) = A_1\hat{a}_+\psi_0 = \frac{A_1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

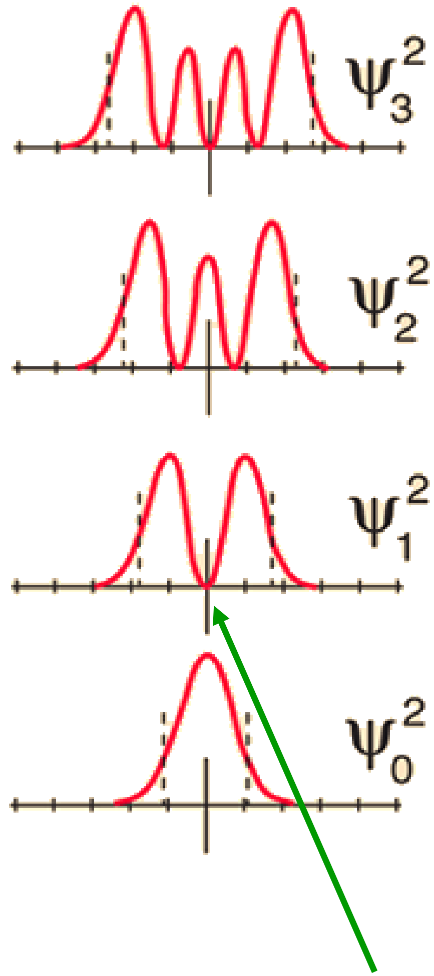
$$= A_1\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}.$$



Odd function







Note: particle can be found outside the classical region

Zero chance at the nodes.

It can be shown, following the textbook, that:

$$\hat{a}_+ \psi_n = \sqrt{n+1}\, \psi_{n+1}$$

$$\hat{a}_- \psi_n = \sqrt{n}\, \psi_{n-1}$$

$$\psi_n = \underbrace{\frac{1}{\sqrt{n!}}}_{A_n} (\hat{a}_+)^n \psi_0$$

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}$$

Orthonormal like we found before for square well.

Example 2.5: find expectation value of V in the *n*th state.

$$\langle V \rangle = \left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{1}{2} m\omega^2 \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx$$

It may be tempting to write ψ_n as a Gaussian with some polynomial in front. However, there is a simpler way.

From $\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x})$ used before, deduce:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-); \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left[(\hat{a}_+)^2 + (\hat{a}_+\hat{a}_-) + (\hat{a}_-\hat{a}_+) + (\hat{a}_-)^2\right]$$

Creates ψ_{n-2} orthog to ψ_n

$$\langle V \rangle = \frac{\hbar\omega}{4} \int \psi_n^* \left[(\hat{a}_+)^2 + (\hat{a}_+\hat{a}_-) + (\hat{a}_-\hat{a}_+) + (\hat{a}_-)^2 \right] \psi_n \, dx.$$

Creates ψ_{n+2} orthog to ψ_n

$$\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \qquad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\hat{a}_-\hat{a}_+\psi_n = \hat{a}_-\sqrt{n+1}\,\psi_{n+1} = \sqrt{n+1}\,\underbrace{\hat{a}_-\psi_{n+1}}_{\sqrt{n+1}\,\psi_n} = (n+1)\psi_n$$

$\hat{a}_+\hat{a}_-$ $\qquad$ $\hat{a}_-\hat{a}_+$

$$\langle V \rangle = \frac{\hbar\omega}{4}(n + n + 1) = \frac{1}{2}\hbar\omega\left(n + \frac{1}{2}\right) = \frac{1}{2}E_n$$