Announcements:

- Average of HW1 was 48/60 points which is 32/40%, within expected range.

- For HW2, in problem 2.7(b) you need some integrals of the “x sin(x)” type. Just find them in tables or use Mathematica.

  Also, use a first page with only your name, and the rest blank. Grader will write provide grade on the second page. Privacy issue.

- Thursday 30 minutes after class, I must give an invited zoom talk. If you send questions Wed late, I may not be able to answer fast. Try to send HW2 related questions not Wed very late, but today afternoon/evening or Wed morning if possible.

- October 4 will be by Zoom.

- Test 2 will be November 1.

(3) Returning to page 28 Ch2 book. How to include time? There are several possible values of E, say E_1, E_2, E_3, …, as found in square well example. For each "allowed" energy, there is a solution of time-indep. Sch. Eq. with its "phase factor"

$$\Psi_1(x,t) = \psi_1(x)e^{-iE_1 t/\hbar}, \quad \Psi_2(x,t) = \psi_2(x)e^{-iE_2 t/\hbar}, \ \ldots.$$

Make a linear combination:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

Statement: any wave function $\Psi(x,t)$ can be written as above. The c_n's are the same as before i.e. time INDEPENDENT. But the linear combination above is NOT a stationary state i.e. $|\Psi(x,t)|^2$ at fixed x, changes with t.

Not in book:

Is this a solution of the time-dep. Sch. Eq. with V(x)?

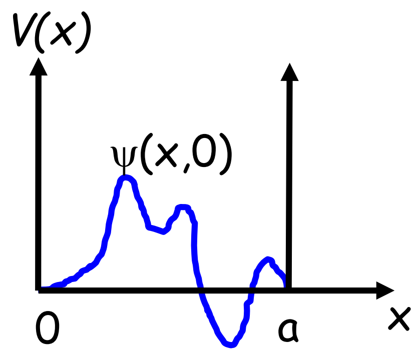
$$\hat{H}\,\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\hat{H}\psi_n(x)}_{E_n\psi_n(x)} e^{-iE_nt/\hbar} \qquad \text{(a)}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)\; \underbrace{i\hbar\frac{\partial}{\partial t}e^{-iE_nt/\hbar}}_{E_n\, e^{-iE_nt/\hbar}} \qquad \text{(b)}$$

(a) is equal to (b) → The linear combination is solution of the time-dependent Sch. Eq., although is not a stationary state.

Final procedure for $\Psi(x,t)$ in square well:

Given an arbitrary $\Psi(x,0)$ -- satisfying the Bound. Conditions -- you are asked for $\Psi(x,t)$.



(1) Find stationary states and energies.

(2) "Somehow" do the integrals for coefficients:

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

(3) Done!

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)}_{\Psi_n(x)} \underbrace{e^{-i(n^2\pi^2\hbar/2ma^2)t}}_{e^{-iE_nt/\hbar}}$$

The procedure is general but of course $\psi_n(x)$ and E_n are diff for diff potentials. This formula is only for square well.

Example 2.1, page 29 book:
Assume you are given at t=0 (typical exam problem!):

Real numbers for simplicity. Assume state normalized i.e. $|c_1|^2 + |c_2|^2 = 1$

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$$

Stationary states sin(nπx/a). NOT any arbitrary function like, say, $e^{-|x|}$

PUNCHLINE: Time dependence is now **trivially** obtained! Eqs 2.16-2.17

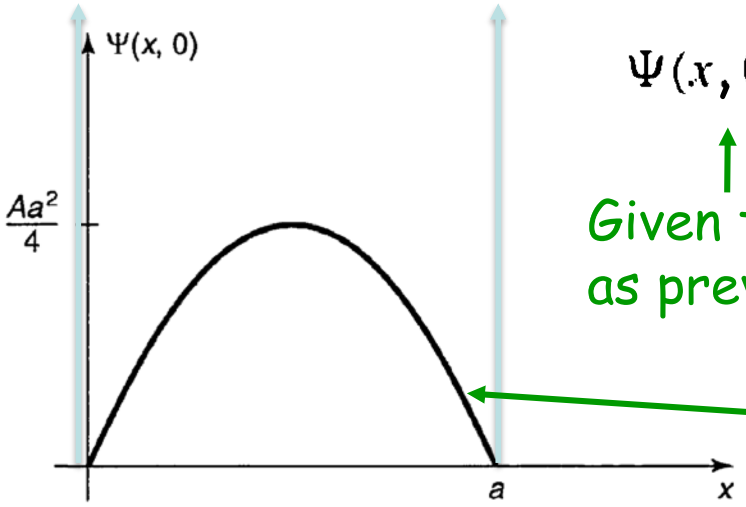
$$\Psi(x,t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar}$$

$$|\Psi(x,t)|^2 = (c_1\psi_1 e^{iE_1t/\hbar} + c_2\psi_2 e^{iE_2t/\hbar})(c_1\psi_1 e^{-iE_1t/\hbar} + c_2\psi_2 e^{-iE_2t/\hbar})$$

$$= c_1^2\psi_1^2 + c_2^2\psi_2^2 + 2c_1c_2\psi_1\psi_2\cos[(E_2 - E_1)t/\hbar].$$

The prob. density is now time dependent even if stationary states are combined ("quantum beat").

Example 2.2: how to apply the recipe (typical exam problem!)



Usual A used to normalize.

$$\Psi(x, 0) = Ax(a - x), \quad (0 \leq x \leq a)$$

Given to you as input, not as easy as previous page problem.

"Even" with respect to middle reflection. Resembles "cosine" ground state but it's not.

Normalize first:

$$1 = \int_0^a |\Psi(x, 0)|^2 \, dx = |A|^2 \int_0^a x^2 (a - x)^2 \, dx \longrightarrow A = \sqrt{\frac{30}{a^5}}$$

Find coefficients c_n:

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sqrt{\frac{30}{a^5}}\, x(a-x)\, dx = 8\sqrt{15}/(n\pi)^3$$

Only *n* odd is nonzero

$\psi_n(x)$ $\qquad$ $\Psi(x,0)$

See integration process in book. Even *n* (i.e. odd functions) gives 0 by symmetry.

You are ready to write the final answer:

$$\Psi(x,t) = \sqrt{\frac{30}{a}} \left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5\ldots} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-in^2\pi^2\hbar t/2ma^2}$$

Often forgotten by students! It contains the time dependence! → $e^{-iE_n t/\hbar}$

We can learn more from this example:

For instance, $\Sigma_n |c_n|^2 = 1$. You can verify that adding, say, the first 10 terms. General reason?

$$1 = \int |\Psi(x,0)|^2 dx = \int \left(\sum_{m=1}^{\infty} c_m \psi_m(x) \right)^* \left(\sum_{n=1}^{\infty} c_n \psi_n(x) \right) dx$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m^* c_n \int \psi_m(x)^* \psi_n(x)\, dx$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_m^* c_n \delta_{mn} = \sum_{n=1}^{\infty} |c_n|^2.$$

This happens only if the given $\Psi(x,0)$ is already normalized to 1.

Thus, $\sum_{n=1}^{\infty} |c_n|^2 = 1$ is equivalent to normalization 1, i.e. the probability of finding the particle "somewhere" inside the well is 100%.

Moreover, $|c_1|^2 = 0.99855...$ i.e. other coefficients are very small. Why? Because $\Psi(x,0)$ closely resembles the ground state! Develop intuition!

If you measure the energy, it will be shown that $|c_n|^2$ is the probability that you will find E_n as result. Here, the chance of measuring E_1 is high.

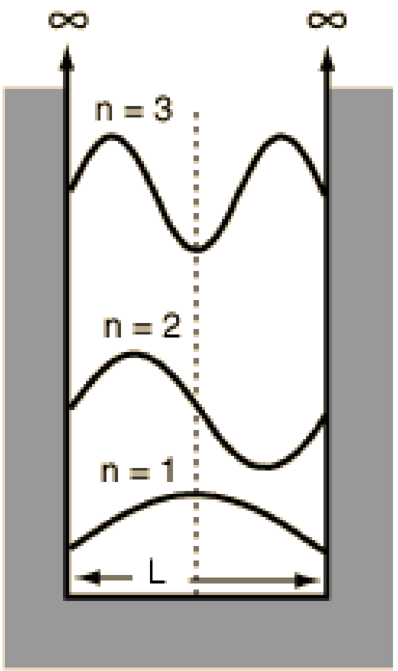
It can also be shown (book p37) that:

$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

In general, for $\langle \hat{O} \rangle$, with $\hat{O}$ any operator, this is not true.

Intuition! Read book: $\langle \hat{H} \rangle$ is only slightly above the ground state energy E_1 compatible with $|c_1|^2 \sim 1$

Summary of infinite square well



x = 0 at left wall of box.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)}_{\psi_n(x)} \underbrace{e^{-i(n^2\pi^2\hbar/2ma^2)t}}_{e^{-iE_nt/\hbar}}$$

The information about the provided initial wave function is hidden in the coefficients via

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$