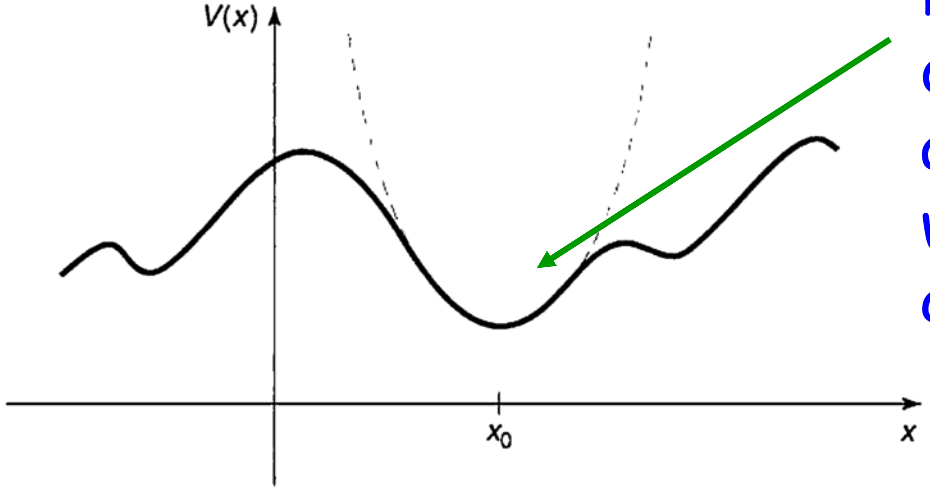
2.3 The harmonic oscillator



Any potential minimum can be approx. by a harmonic oscillator, if oscillations are small. Thus, this is widely used! Even E&M is quantized this way.

We wish to study the harmonic oscillator from the QM perspective:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \underbrace{\frac{1}{2}m\omega^2 x^2}\psi = E\psi$$

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

Two methods will be used to arrive to the same answer:

Algebraic method (uses $\hat{a}_+$ and $\hat{a}_-$ operators)

Analytic method (uses polynomials)

Warning: the symbol ^ will sometimes appear, sometimes not. You have to develop the ability to judge if a quantity is an operator or not.

Preliminary exercise: commutation relations

We define the commutator between operators $\hat{A}$ and $\hat{B}$ as:

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Suppose $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$. Thus, we want $[\hat{x}, \hat{p}] = (\hat{x}\hat{p} - \hat{p}\hat{x})$.
This is NOT zero because $\hat{p}$ is the derivative operator.

To know its value you need a general "test function" *f(x)*.

$$[\hat{x}, \hat{p}]f(x) = \left[x\frac{\hbar}{i}\frac{d}{dx}(f) - \frac{\hbar}{i}\frac{d}{dx}(xf)\right] = \frac{\hbar}{i}\left(x\frac{df}{dx} - x\frac{df}{dx} - f\right) = i\hbar f(x)$$

$$[\hat{x}, \hat{p}] = i\hbar$$

In general, operators do not commute!

Finding the commutator of two given operators is a typical test problem.

2.3.1 Algebraic method

Sum of squares

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \longrightarrow \frac{1}{2m}[\hat{p}^2+(m\omega\hat{x})^2]\psi=E\psi$$

$$\hat{p}\equiv(\hbar/i)d/dx$$

For u,v real numbers $u^2+v^2=(iu+v)(-iu+v)$ but not for operators.

This factor is merely for future convenience.

By analogy try $\hat{a}_\pm\equiv\frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p}+m\omega\hat{x})$ and cross fingers.

$$\hat{a}_-\hat{a}_+=\frac{1}{2\hbar m\omega}(i\hat{p}+m\omega\hat{x})(-i\hat{p}+m\omega\hat{x})=\frac{1}{2\hbar m\omega}[\hat{p}^2+(m\omega\hat{x})^2-im\omega(\hat{x}\hat{p}-\hat{p}\hat{x})]$$

$$\hat{a}_-\hat{a}_+=\frac{1}{2\hbar m\omega}[\hat{p}^2+(m\omega\hat{x})^2]-\frac{i}{2\hbar}[\hat{x},\hat{p}]=\frac{1}{\hbar\omega}\hat{H}+\frac{1}{2} \Longrightarrow \hat{H}=\hbar\omega\left(\hat{a}_-\hat{a}_+-\frac{1}{2}\right)$$

$$[\hat{x},\hat{p}]=i\hbar$$

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

Or, using the other order, we get (left as exercise):

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

Theorem: if ψ satisfies $\hat{H}\psi = E\psi$, then $\hat{H}(\hat{a}_+\psi) = (E + \hbar\omega)(\hat{a}_+\psi)$

$$\hat{H}(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{1}{2}\hat{a}_+\right)\psi$$

$$= \hbar\omega\hat{a}_+\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)\psi = \hat{a}_+\left[\hbar\omega\left(\hat{a}_-\hat{a}_+ + 1 - \frac{1}{2}\right)\psi\right]$$

$$= \hat{a}_+(\hat{H} + \hbar\omega)\psi = \hat{a}_+(E + \hbar\omega)\psi = (E + \hbar\omega)(\hat{a}_+\psi).$$

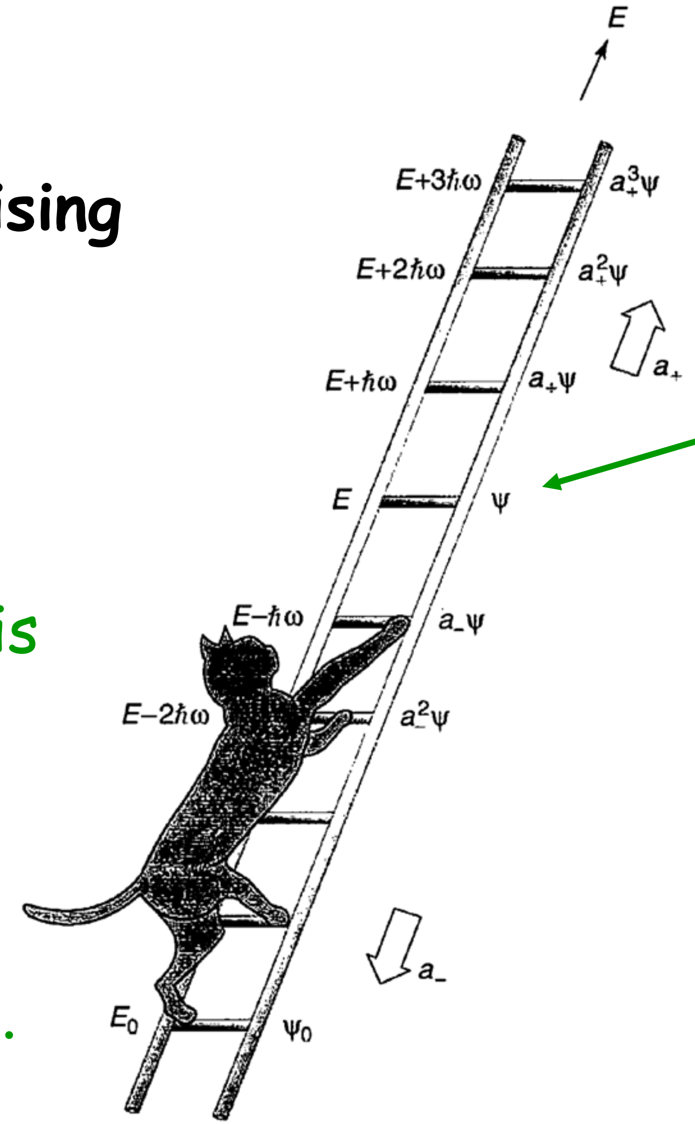
If I know one solution, I know another solution, etc. ...

Another theorem: if ψ satisfies $\hat{H}\psi = E\psi$, then $\hat{H}(\hat{a}_-\psi) = (E - \hbar\omega)(\hat{a}_-\psi)$ *(left as exercise)*

$\hat{a}_+$ is the **raising** operator

Equally spaced energies! This is why it works.

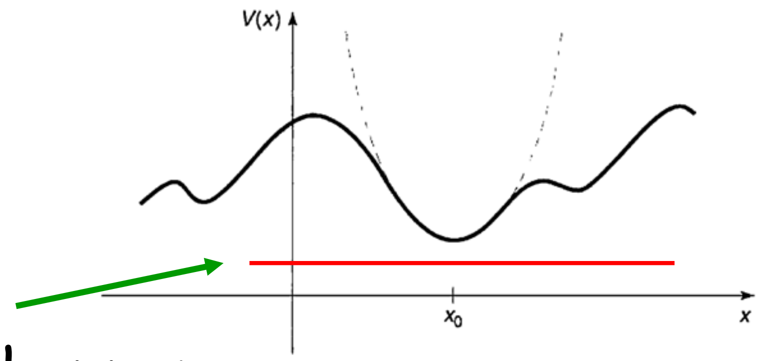
Not general: valid for oscillators only.



If I am given one solution, I get infinite more.

$\hat{a}_-$ is the **lowering** operator

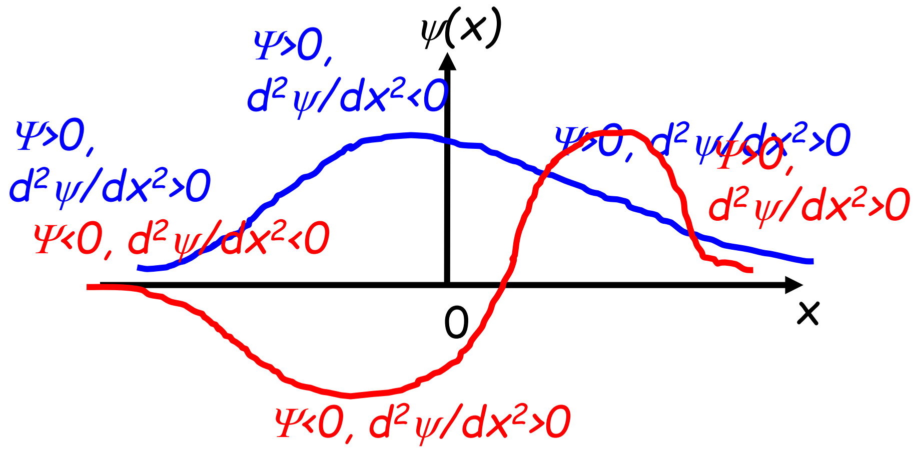
However, there is a problem:
the energy cannot continue going down!



<u>Theorem</u>: *E* less than *V(x)* cannot happen

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \longrightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$$

If this is >0 for all x, as in the incorrect sketch above, then ψ and $d^2\psi/dx^2$ must have same sign. Integrable functions do not satisfy this constraint.



Because the energy cannot continue going down forever, the chain down must stop…