Eventually once true ground state ψ_0 is reached, then $\hat{a}_-\psi_0$ must be 0. We can use this condition to find ψ_0 .

$$\underbrace{\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx}+m\omega x\right)}_{\hat{a}_-}\psi_0=0 \longrightarrow \frac{d\psi_0}{dx}=-\frac{m\omega}{\hbar}x\psi_0$$

$$\int\frac{d\psi_0}{\psi_0}=-\frac{m\omega}{\hbar}\int x\,dx \quad\Rightarrow\quad \ln\psi_0=-\frac{m\omega}{2\hbar}x^2+\text{constant},$$

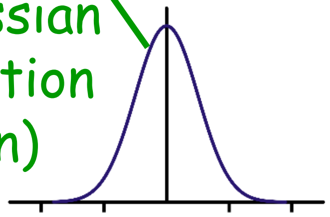
$$\boxed{\psi_0(x)=Ae^{-\frac{m\omega}{2\hbar}x^2}.}$$

After normalization:
(left as exercise, use Gaussian integrals in back cover of book)

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Check!

Gaussian function (even)



What is the energy E_0?

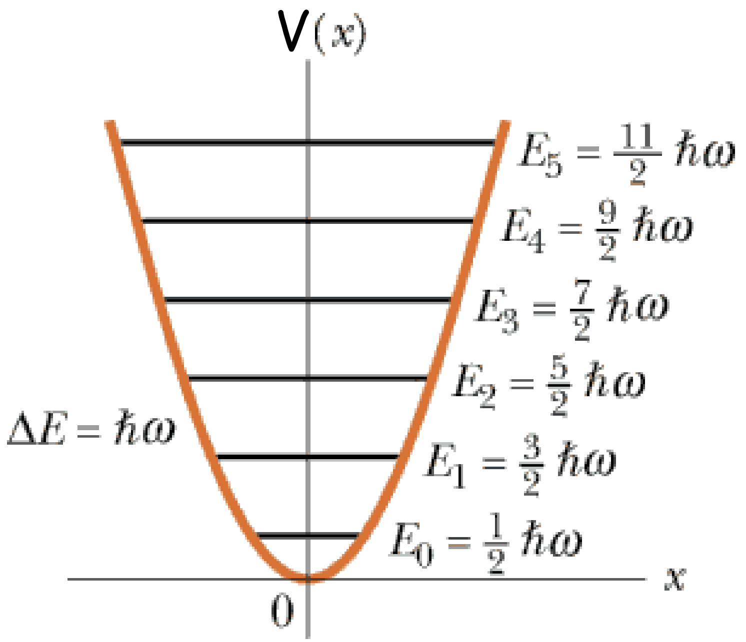
$$\hbar\omega(\underbrace{\hat{a}_+\overbrace{\hat{a}_-}^{=0} + 1/2)}_{\hat{H}}\psi_0 = E_0\psi_0$$

$\hat{H}$ (Hamiltonian); BTW note how easy was the calculation of E_0.

Then:

$$E_0 = \frac{1}{2}\hbar\omega$$

>0 as expected. Deep meaning: QM has "Zero point energy". The harmonic oscillators, or any other QM problem, are never still!

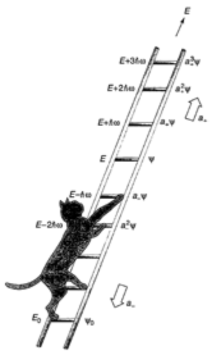


Equally spaced levels

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

Solutions are

$$\psi_n(x) = A_n(\hat{a}_+)^n\psi_0(x)$$

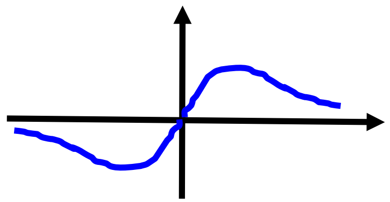


Example 2.4: construct state 1

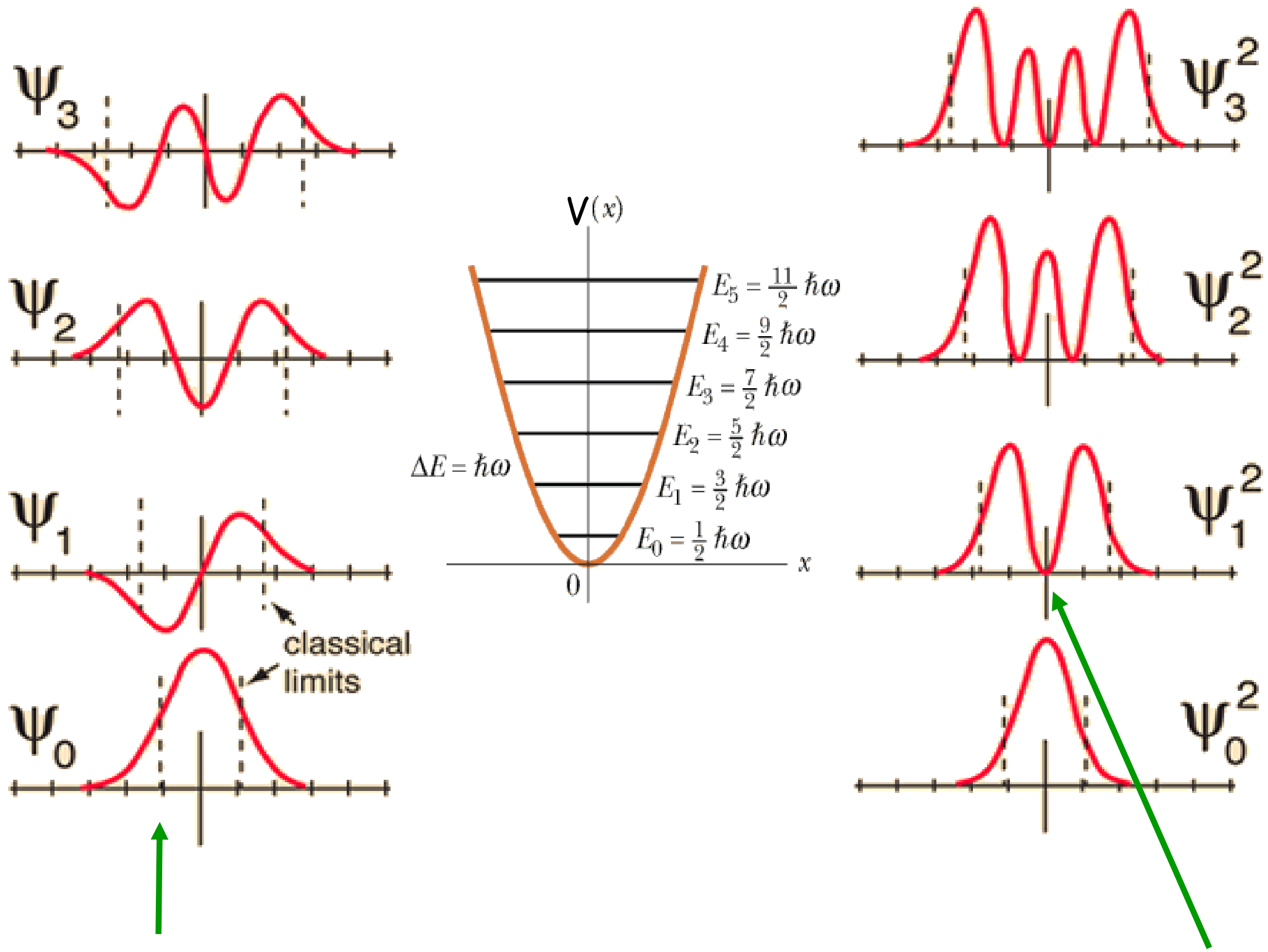
Brings down an x

$$\psi_1(x) = A_1\hat{a}_+\psi_0 = \frac{A_1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A_1\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}.$$



Odd function



Note: particle can be found outside the classical region

~Zero chance near nodes.

It can be shown, following the textbook, that:

$$\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$\hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\psi_n = \underbrace{\frac{1}{\sqrt{n!}}}_{A_n}(\hat{a}_+)^n\psi_0$$

$$\int_{-\infty}^{\infty}\psi_m^*\psi_n\,dx = \delta_{mn}$$

Orthonormal, like we found before for square well.

Example 2.5: find expectation value of V in the n-th state.

$$\langle V \rangle = \left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{1}{2} m\omega^2 \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx$$

It may be tempting to write ψ_n as a Gaussian with some polynomial in front. However, there is a simpler way.

From $\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}} (\mp i\hat{p} + m\omega\hat{x})$ used before, deduce:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-); \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} (\hat{a}_+ - \hat{a}_-)$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega} \left[(\hat{a}_+)^2 + (\hat{a}_+\hat{a}_-) + (\hat{a}_-\hat{a}_+) + (\hat{a}_-)^2 \right]$$

Creates ψ_{n-2} orthog to ψ_n

$$\langle V\rangle = \frac{\hbar\omega}{4}\int \psi_n^*\left[(\hat{a}_+)^2 + (\hat{a}_+\hat{a}_-) + (\hat{a}_-\hat{a}_+) + (\hat{a}_-)^2\right]\psi_n\,dx.$$

Creates ψ_{n+2} orthog to ψ_n

$$\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\hat{a}_-\hat{a}_+\psi_n = \hat{a}_-\sqrt{n+1}\,\psi_{n+1} = \sqrt{n+1}\,\underbrace{\hat{a}_-\psi_{n+1}}_{\sqrt{n+1}\,\psi_n} = (n+1)\psi_n$$

$\hat{a}_+\hat{a}_-$ $\hat{a}_-\hat{a}_+$

$$\langle V\rangle = \frac{\hbar\omega}{4}(n + n + 1) = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) = \frac{1}{2}E_n$$

About HW3

(2.10) and (2.11)

Using $\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n \psi_0(x)$, and $\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})$,

and $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$;

you move the entire problem into "x space" and integrate, etc.

(2.12)

Following example 2.5 book, you keep wave functions in terms of $\hat{a}_+$ operators and, instead, transform, for example, $\hat{x}^2$ or $\hat{p}^2$ in the language of $\hat{a}$ and $\hat{a}_+$. Then, find expectation values.

Often this procedure is easier.