Since we had so much fun, let us redo the harmonic oscillator! ☺ : 2.3.2 The Analytic Method.

multiply by $-2/\hbar\omega$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$$

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi \qquad K \equiv \frac{2E}{\hbar\omega}$$

Now we follow traditional approach: derivatives on one side, the rest on the other.

At large ξ: $$\frac{d^2\psi}{d\xi^2} \approx \xi^2\psi \longrightarrow \psi(\xi) \approx Ae^{-\xi^2/2}$$

Valid only at large ξ. Also note that the "+" exponential is not normalizable.

Then, propose $\psi(\xi) = h(\xi)e^{-\xi^2/2}$

same exponential as found before

"milder" than exponential, like a polynomial

$$\frac{d\psi}{d\xi} = \left(\frac{dh}{d\xi} - \xi h\right) e^{-\xi^2/2}$$

Sch. Eq. previous page

$$\frac{d^2\psi}{d\xi^2} = \left(\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (\xi^2 - 1)h\right) e^{-\xi^2/2} = (\xi^2 - K)\psi$$

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0.$$

"New" Sch. Eq.
At first sight looks worse than before!

Try a power series or polynomial (a are coefficients; should not be confused with previous operators $\hat{a}$):

$$h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{j=0}^{\infty} a_j \xi^j$$

$$-2\xi\left(\frac{dh}{d\xi}\right) = -2\xi\left(a_1 + 2a_2\xi + 3a_3\xi^2 + \cdots\right) = -2\xi\left(\sum_{j=0}^{\infty} j a_j \xi^{j-1}\right)$$

$$\frac{d^2h}{d\xi^2} = 2a_2 + 2\cdot 3a_3\xi + 3\cdot 4a_4\xi^2 + \cdots = \sum_{j=0}^{\infty}(j+1)(j+2)a_{j+2}\xi^j$$

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0.$$

$$\sum_{j=0}^{\infty}\left[(j+1)(j+2)a_{j+2} - 2ja_j + (K-1)a_j\right]\xi^j = 0$$

$$(j+1)(j+2)a_{j+2} - 2ja_j + (K-1)a_j = 0$$

$$a_{j+2} = \frac{(2j+1-K)}{(j+1)(j+2)} a_j$$

j = 0, 1, 2, 3, … Even (0,2,4,…) are separated from odd (1,3,5,…).

However, this series cannot go on forever.
It must terminate and become a polynomial.

Reason: at large j, $a_{j+2} = 2/j\ a_j$, $a_{j+4} = (2/j+2)(2/j)\ a_j$, … .
At large j, $a_j \sim 1/(j/2)!$, thus

$$\sum \frac{1}{(j/2)!} \xi^j \approx_{j/2=n} \sum \frac{1}{n!} \xi^{2n} \approx e^{\xi^2}$$

$$e^x = \sum_n \frac{1}{n!} x^n$$

which diverges at large ξ, thus it is not normalizable.

There must be an "n" beyond which $a_{n+2}=0$ …
both for the even and odd sectors.

But the "n" is not unique, can be any integer.
For each n that we choose, we will find a E_n.

$$(n+1)(n+2)\underbrace{a_{n+2}}_{=0} \underbrace{- 2na_n + (K-1)a_n}_{} = 0$$

Implies $-2n + (K-1) = 0$ or $K = 2n+1$

Recalling $K \equiv \frac{2E}{\hbar\omega}$, we arrive to $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$
that we know is correct. Check!

Note: I am not sure what the author tries to say with Fig. 2.6. I suggest to ignore it.

Solutions? I will have one solution with only a_0, one with only a_0 and a_2, one with only a_0, a_2, and a_4, …, only one solution with a_1, one with only a_1 and a_3, one with only a_1, a_3, and a_5, …,

n=0 $$\psi_0(\xi) = a_0 e^{-\xi^2/2}$$

n=1 $$\psi_1(\xi) = a_1 \xi e^{-\xi^2/2}$$

n=2 $$\psi_2(\xi) = a_0(1 - 2\xi^2) e^{-\xi^2/2}$$

$$a_{j+2} = \frac{(2j+1-K)}{(j+1)(j+2)} a_j \quad j=0 \quad \longrightarrow \quad a_2 = \frac{(1-K)}{2} a_0$$

For n=2, $K = 2n+1 = 5$. Thus, $a_2 = (1-5)/2\ a_0 = -2\ a_0$

These are the same solutions found before with the raising and lowering operators! In general:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

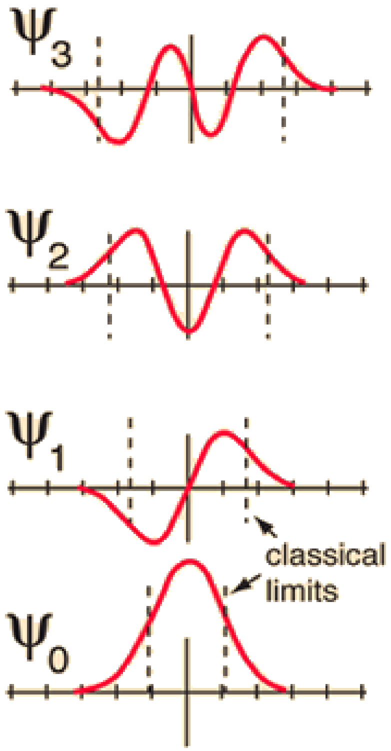
Identical to previously found solutions (no need to check) $\psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n \psi_0$

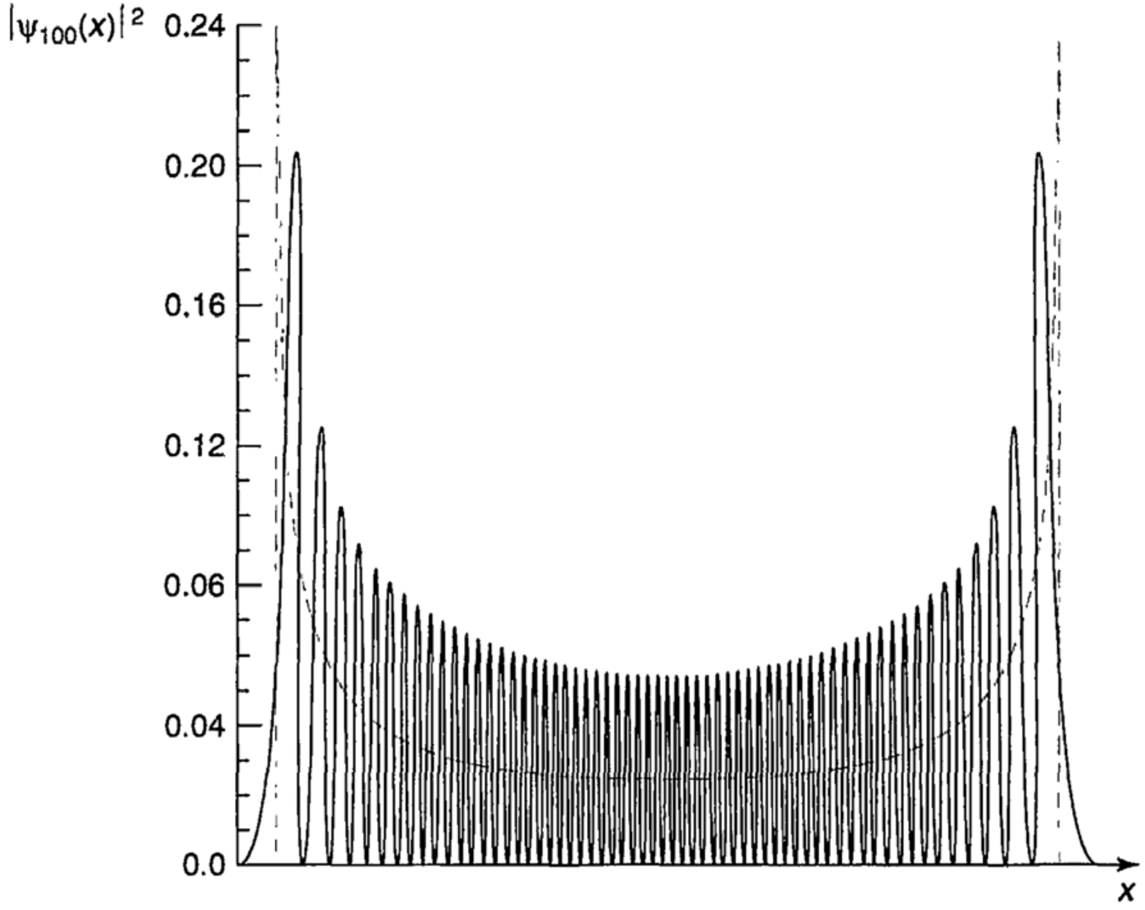
$$H_0 = 1,$$
$$H_1 = 2\xi,$$
$$H_2 = 4\xi^2 - 2,$$
$$H_3 = 8\xi^3 - 12\xi,$$
$$H_4 = 16\xi^4 - 48\xi^2 + 12,$$
$$H_5 = 32\xi^5 - 160\xi^3 + 120\xi.$$

Hermite polynomials. H_n has n nodes. They are even and odd functions.

Ermeet: french

Hermite : english





Amazingly at large *n*, the results become similar to the classical result for a harmonic oscillator (dashed) (general result for all problems of QM)