There is more we can learn from this example.

For instance, $\Sigma_n |c_n|^2 = 1$. You can verify that adding, say, the first 10 terms. Reason?

$$1 = \int |\Psi(x,0)|^2 dx = \int \left(\sum_{m=1}^{\infty} c_m \psi_m(x) \right)^* \left(\sum_{n=1}^{\infty} c_n \psi_n(x) \right) dx$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m^* c_n \int \psi_m(x)^* \psi_n(x)\, dx$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_m^* c_n \delta_{mn} = \sum_{n=1}^{\infty} |c_n|^2.$$

This happens only if the given $\Psi(x,0)$ is already normalized to 1.

Thus, $\sum_{n=1}^{\infty} |c_n|^2 = 1$ is equivalent to normalization 1, which is probability of finding the particle somewhere in the well 100%.

Moreover, $|c_1|^2 = 0.99855...$ implying other coefficients are very small. Why? Because $\Psi(x,0)$ resembles the ground state! Develop intuition!

If you measure the energy, it will be shown that $|c_n|^2$ is the probability that you will find E_n as result. Here, the chance of measuring E_1 is high.

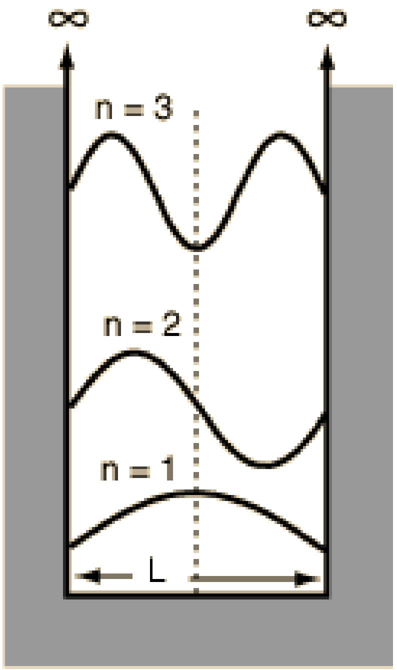
It can also be shown (book p37) that:

$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

In general for $\langle \hat{O} \rangle$, with $\hat{O}$ any operator, this is not true.

Intuition! Read book: $\langle \hat{H} \rangle$ is only slightly above the ground state energy E_1 compatible with $|c_1|^2 \sim 1$

Summary of infinite square well



x = 0 at left wall of box.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

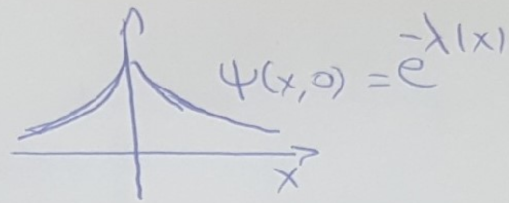
$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$\psi_n(x)$ $\quad$ $e^{-iE_n t/\hbar}$

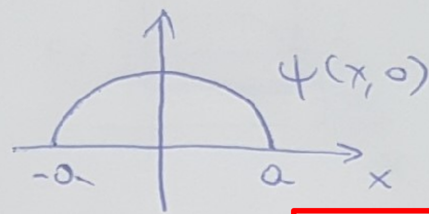
$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$

Comments about HW1 after grading:

Problem 1.5



Problem 1.16



By mere symmetry $\langle x \rangle = 0$.

If you found $\langle x \rangle \neq 0$ you must realize there was a math error.

If you found $\sigma_x = 0$, there is also a math error because the wave function has an obvious size (not a δ-function).

If you found $\langle x^2 \rangle = a^2 + \frac{1}{3} a^3$ in 1.16 then you must realize there is an error. a^2 has the correct units but a^3 not. It is like adding $m^2 + m^3$ (m = meters).

Problem 1.7

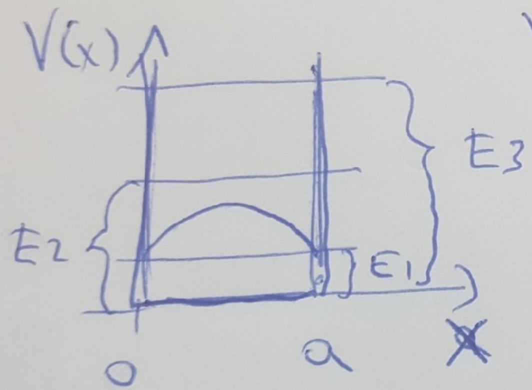
$$\frac{dp}{dt} = F = -\frac{\partial V(x)}{\partial x} \longrightarrow \frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle$$

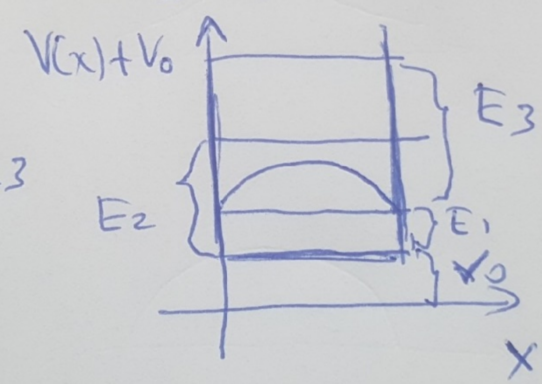
classical quantum

Expectation values behave classically.

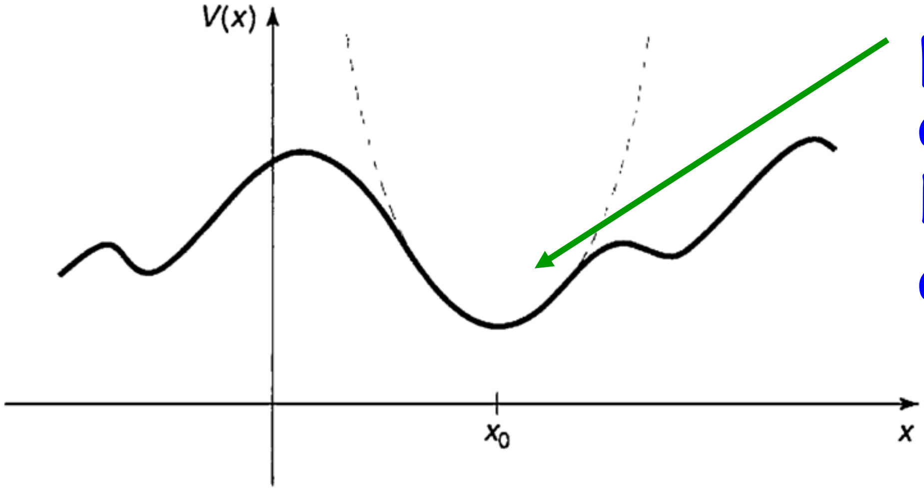
Problem 1.8

A shift by a constant of V(x) does not change the physics





2.3 The harmonic oscillator



Any minimum of a potential can be approximated by a harmonic oscillator as long as oscillations are small.

We wish to study the harmonic oscillator from the QM perspective:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$$

$$V(x)=\frac{1}{2}m\omega^2x^2$$

Two methods will be used to arrive to the same answer:

Algebraic method (uses $\hat{a}_+$ and $\hat{a}_-$ operators)

Analytic method (uses polynomials)

Warning: the symbol ^ will sometimes appear, sometimes not. You have to develop the ability to judge if a quantity is an operator or not.

Preliminary exercise: commutation relations

We define the commutator between operators $\hat{A}$ and $\hat{B}$ as:

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Suppose $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$. Thus, we want $[\hat{x}, \hat{p}] = (\hat{x}\hat{p} - \hat{p}\hat{x})$.
This is NOT zero because $\hat{p}$ is the derivative operator.

To know its value you need a general "test function" $f(x)$.

$$[\hat{x}, \hat{p}]f(x) = \left[x\frac{\hbar}{i}\frac{d}{dx}(f) - \frac{\hbar}{i}\frac{d}{dx}(xf)\right] = \frac{\hbar}{i}\left(x\frac{df}{dx} - x\frac{df}{dx} - f\right) = i\hbar f(x)$$

$$[\hat{x}, \hat{p}] = i\hbar$$

In general, operators do not commute!

Finding the commutator of two given operators is a typical test problem.

2.3.1 Algebraic method

Sum of squares

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \longrightarrow \frac{1}{2m}[\hat{p}^2+(m\omega\hat{x})^2]\psi=E\psi$$

$$\hat{p}\equiv(\hbar/i)d/dx$$

For u,v real numbers $u^2+v^2=(iu+v)(-iu+v)$ but not for operators.

This factor is merely for future convenience.

By analogy try $\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p}+m\omega\hat{x})$ and cross fingers.

$$\hat{a}_-\hat{a}_+=\frac{1}{2\hbar m\omega}(i\hat{p}+m\omega\hat{x})(-i\hat{p}+m\omega\hat{x})=\frac{1}{2\hbar m\omega}[\hat{p}^2+(m\omega\hat{x})^2-im\omega(\hat{x}\hat{p}-\hat{p}\hat{x})]$$

$$\hat{a}_-\hat{a}_+=\frac{1}{2\hbar m\omega}[\hat{p}^2+(m\omega\hat{x})^2]-\frac{i}{2\hbar}[\hat{x},\hat{p}]=\frac{1}{\hbar\omega}\hat{H}+\frac{1}{2} \quad \Rightarrow \quad \hat{H}=\hbar\omega\left(\hat{a}_-\hat{a}_+-\frac{1}{2}\right)$$

$$[\hat{x},\hat{p}]=i\hbar$$

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

Or, using the other order, we get (left as exercise):

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

Theorem: if ψ satisfies $\hat{H}\psi = E\psi$, then $\hat{H}(\hat{a}_+\psi) = (E + \hbar\omega)(\hat{a}_+\psi)$

$$\hat{H}(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{1}{2}\hat{a}_+\right)\psi$$

$$= \hbar\omega\hat{a}_+\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)\psi = \hat{a}_+\left[\hbar\omega\left(\hat{a}_-\hat{a}_+ + 1 - \frac{1}{2}\right)\psi\right]$$

$$= \hat{a}_+(\hat{H} + \hbar\omega)\psi = \hat{a}_+(E + \hbar\omega)\psi = (E + \hbar\omega)(\hat{a}_+\psi).$$

If I know one solution, I know another solution …