Chapter 9: The WKB Approximation

Back to time independent problems. WKB stands for Wentzel, Kramers, Brillouin.

WKB is a technique to obtain approximate solutions to time independent problems, mainly in 1D or where only "r" matters in 3D.

Main intuitive idea: suppose you have a potential V(x)=V totally constant, no imperfections. Then, the solution if $E > V(x)$ is:

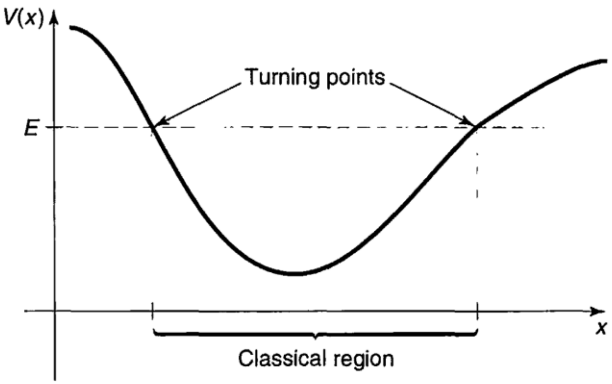
$$\psi(x) = Ae^{\pm ikx} \qquad k \equiv \sqrt{2m(E-V)}/\hbar \qquad \lambda = 2\pi/k$$

Of course, here *A* is constant, *k* is constant, λ is constant.

However, a perfectly flat potential is unlikely. Suppose V(x) is "nearly" flat but changes very slowly with x, i.e. over distances much larger than λ. Then, the solution cannot be too different: A, k,… will now be smooth slowly varying functions of x.

9.1: The "Classical" Region

Let us first consider the case E>V(x), i.e. the classical region. First, we will not make any approximation and find exact equations for amplitude and phase. Then, we will make the WKB approximation.



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \qquad E > V(x)$$

Exactly, this can be written $\dfrac{d^2\psi}{dx^2} = -\dfrac{p^2}{\hbar^2}\psi$ where $p(x) \equiv \sqrt{2m[E - V(x)]}$

Propose $\psi(x) = A(x)e^{i\phi(x)}$, which is generic for any wave function. Here both *A(x)* and ϕ *(x)* are real functions, dependent on x.

$$\frac{d\psi}{dx} = (A' + iA\phi')e^{i\phi}$$

$$\frac{d\psi}{dx} = (A' + iA\phi')e^{i\phi} \longrightarrow \frac{d^2\psi}{dx^2} = [A'' + 2iA'\phi' + iA\phi'' - A(\phi')^2]e^{i\phi}$$

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi \longrightarrow A'' + 2iA'\phi' + iA\phi'' - A(\phi')^2 = -\frac{p^2}{\hbar^2}A$$

$$p(x) \equiv \sqrt{2m[E - V(x)]}$$

Real part:

$$A'' - A(\phi')^2 = -\frac{p^2}{\hbar^2}A$$

$$A'' = A\left[(\phi')^2 - \frac{p^2}{\hbar^2}\right]$$

Cannot be solved unless we assume A″~0, i.e. amplitude varies slowly with x.

Imaginary part:

$$2A'\phi' + A\phi'' = 0$$

$$\left(A^2\phi'\right)' = 0$$

$$A^2\phi' = C^2$$

Exact: $$A = \frac{C}{\sqrt{\phi'}}$$

Again, the two exact eqs. are:

$$A'' = A\left[(\phi')^2 - \frac{p^2}{\hbar^2}\right] \qquad A = \frac{C}{\sqrt{\phi'}}$$

If A"~0 because V(x) changes slowly, then:

$$(\phi')^2 = \frac{p^2}{\hbar^2}$$

$$\phi(x) = \pm\frac{1}{\hbar}\int p(x)\,dx$$

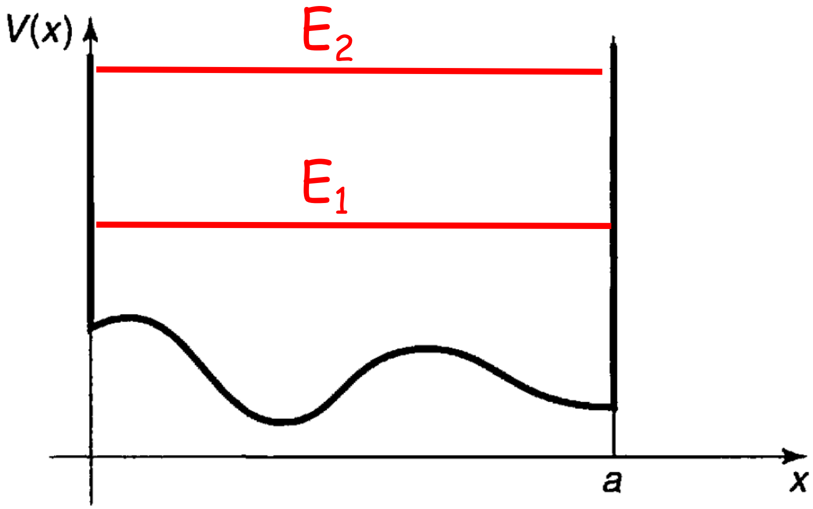
We started with $\psi(x) = A(x)e^{i\phi(x)}$ then we arrive to:

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}} e^{\pm\frac{i}{\hbar}\int p(x)\,dx}$$

This is the WKB approximation to the wave function.

Note $\phi(x)$ is an indefinite integral i.e. x dependent.
We will need boundary conditions.

Example 9.1: Potential well with two vertical walls.



Assume E > V(x) for all values of x (this may or may not be correct, we just assume it).

We found before:

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x)\, dx}$$

In general, we have to make a linear combination:

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}} \left[C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)} \right]$$

where $$\phi(x) = \frac{1}{\hbar} \int_0^x p(x')\, dx'$$

Repeating $\psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)}\right]$

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[C_1 \sin\phi(x) + C_2 \cos\phi(x)\right]$$

Boundary conditions:

(1) $\psi(x) = 0$ at $x = 0$

This means $C_2 = 0$

(2) $\psi(x) = 0$ at $x = a$

This means $\phi(a) = n\pi \quad (n = 1, 2, 3, \ldots)$

$$\phi(a) = n\pi \quad (n = 1, 2, 3, \ldots)$$

means

$$\int_0^a p(x)\, dx = n\pi\hbar$$

$$\int_0^a \sqrt{2m[E - V(x)]}\; dx = n\pi\hbar$$

where *E* is the unknown for each "*n*".

The integral can be done analytically and an equation for *E* will be found, or we can find *E* numerically.

Test: If V(x)=0 inside the well, then of course we know the answer.

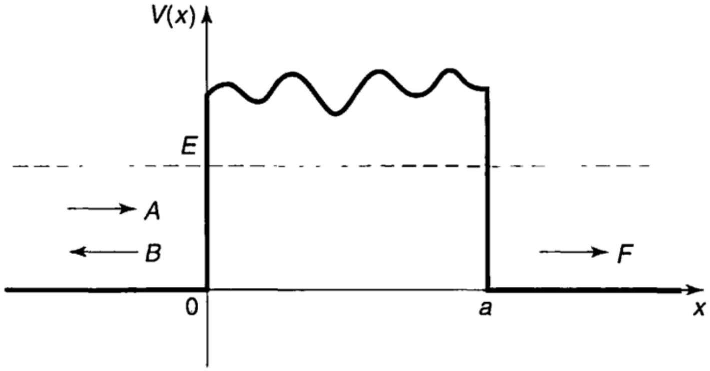
$$\int_0^a \sqrt{2m[E - V(x)]} = \int_0^a \sqrt{2m\ E}\ dx = n\pi\hbar$$

$$\sqrt{2m\ E}\ a = n\pi\hbar$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

which is the exact result.

9.2: "Tunneling"



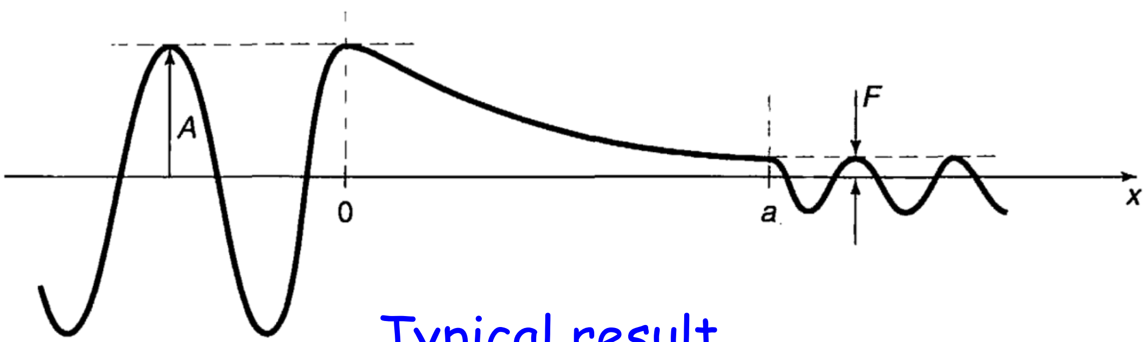
Now consider regions that are NOT classical i.e. E<V(x).

We can repeat all the same and we find:

Indefinite integral, i.e. x dependent

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar}\int |p(x)|\,dx}$$

Note: no "*i*" in phase and |..| in *p(x)*



Typical result

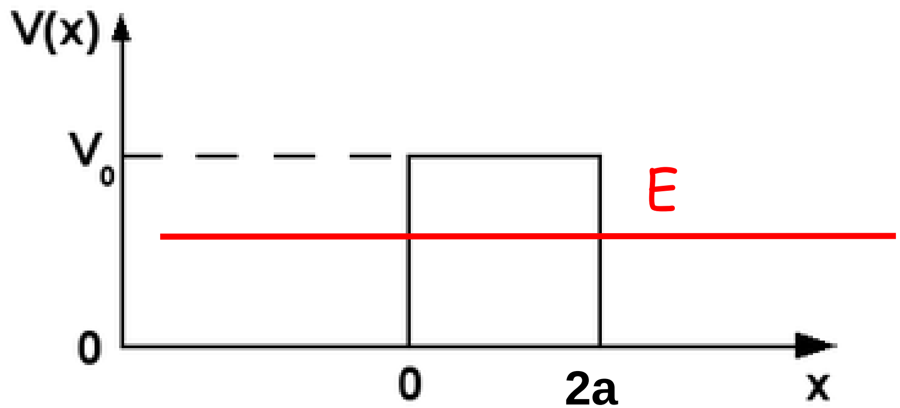
$$\frac{|F|}{|A|} \sim e^{-\frac{1}{\hbar}\int_0^a |p(x')|\,dx'}$$

The result of previous page is a general result:

$$\frac{|F|^2}{|A|^2} \sim e^{-\frac{1}{\hbar}2\int_0^a |p(x')|\,dx'}$$

$$T \cong e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$

Example,
Problem 9.3 of HW9.
Yes, I am giving you the solution ☺.



This is the **exact** result from Ch. 2, problem 2.33, page 75:

$$T_{\text{exact}} = \frac{1}{1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma}$$

where

$$\gamma = \frac{2a}{\hbar}\sqrt{2m(V_0-E)}$$

We pretend we do not know the exact result and try to use the WKB approximation:

General WKB approx. for tunneling through barrier of width "a" is:

$$T \cong e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$

Width of barrier is 2a here:

$$\gamma = \frac{1}{\hbar}\int |p(x)|\,dx = \frac{1}{\hbar}\int_0^{2a} \sqrt{2m(V_0 - E)}\,dx = \frac{2a}{\hbar}\sqrt{2m(V_0 - E)}$$

Then, WKB prediction for tunneling is then:

$$T \approx e^{-4a\sqrt{2m(V_0-E)}/\hbar}$$

$$T_{\text{exact}} = \frac{1}{1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma} \quad \Bigg| \quad \sinh\gamma = \tfrac{1}{2}(e^{\gamma} - e^{-\gamma}) \approx \tfrac{1}{2}e^{\gamma} \quad \Bigg| \quad \sinh^2\gamma \approx \tfrac{1}{4}e^{2\gamma}$$

$$T_{\text{exact}} \approx \frac{1}{1 + \frac{V_0^2}{16E(V_0-E)}e^{2\gamma}} \approx \left\{\frac{16E(V_0 - E)}{V_0^2}\right\} e^{-2\gamma}$$

$$\gamma_{\text{exact}} = \frac{2a}{\hbar}\sqrt{2m(V_0 - E)}$$

Dominant exponential, i.e. 2γ, is exactly reproduced by WKB.