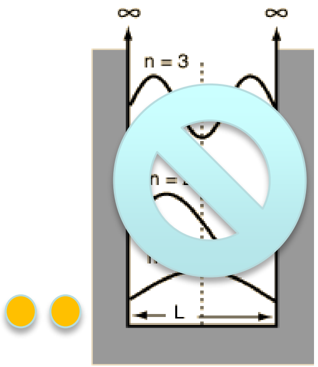
(3) If particles are fermions:

Because temporarily I am forgetting about the spin, then I cannot place 2 particles in $n_1 = 1$.

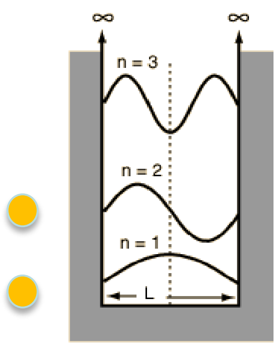


x = 0 at left wall of box.

Then, the true ground state (E=5K) becomes:

$$\psi(x_1,x_2) = \frac{\sqrt{2}}{a}\left[\sin(\pi x_1/a)\sin(2\pi x_2/a) - \sin(2\pi x_1/a)\sin(\pi x_2/a)\right]$$

$$\hat{P}\,\psi(x_1,x_2) = \psi(x_2,x_1) = -\psi(x_1,x_2)$$



x = 0 at left wall of box.

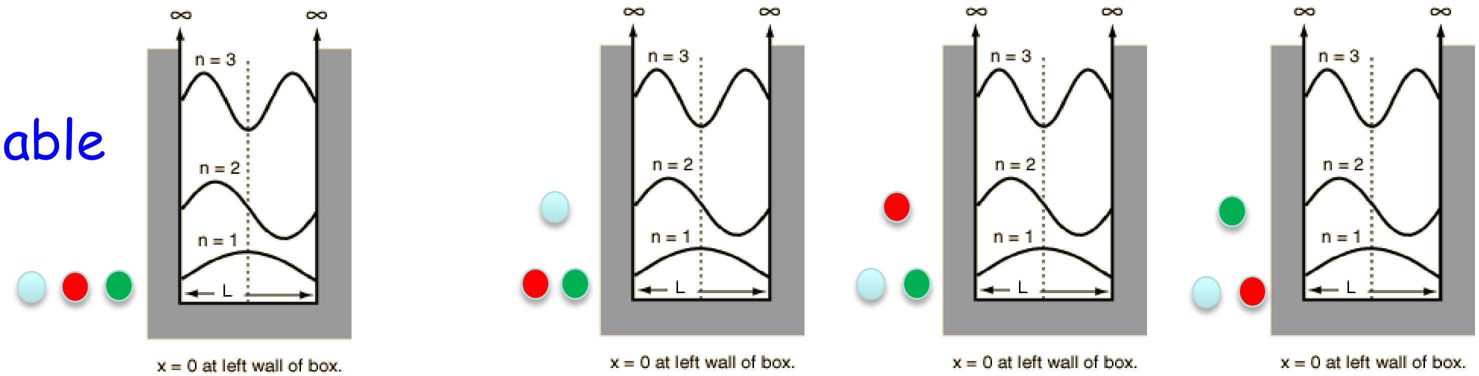
In summary (if spin is not included):

	distinguishable	bosons	fermions
E=2K	1 state	1 state	0 state
E=5K	2 states	1 state	1 state
etc			

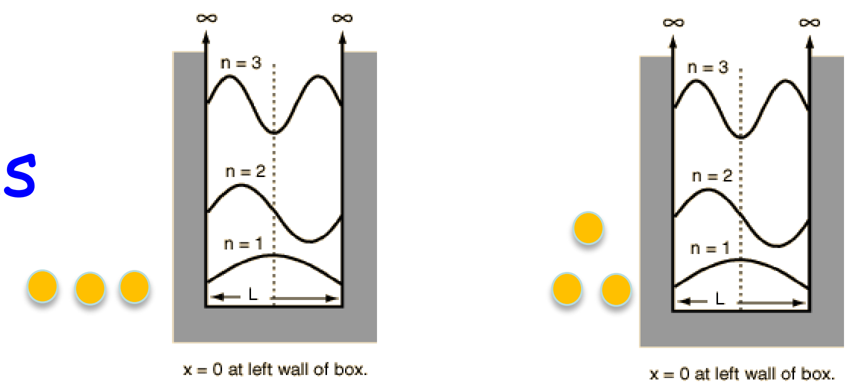
These differences has profound implications for many-body physics (condensed matter, nuclear, …) and for thermodynamics

Three particles? (spin still not included)

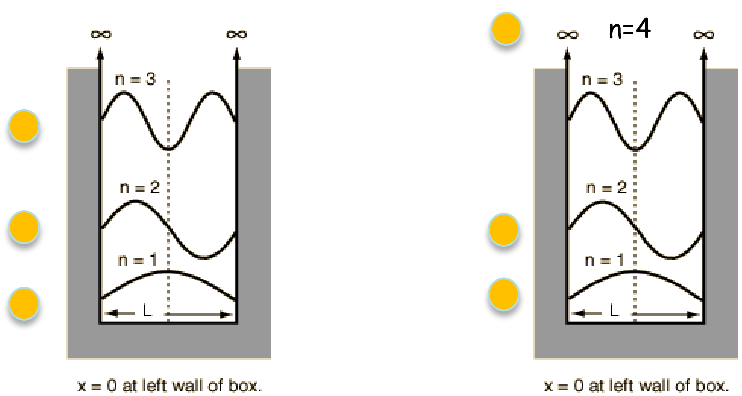
distinguishable



bosons



fermions



5.1.2 Exchange Forces

This is an "effective force", not a real one.

It is another of the consequences of having symmetric or antisymmetric combinations.
Consider 1D and two different states *a* and *b*.

distinguishable

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2)$$

bosons

$$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)]$$

fermions

$$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)]$$

Consider the typical "distance" between particles, via the quantity:

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

Distance in 1D

distinguishable

$$\langle x_1^2 \rangle = \int x_1^2 |\psi_a(x_1)|^2 \, dx_1 \int |\psi_b(x_2)|^2 \, dx_2 = \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \int |\psi_a(x_1)|^2 \, dx_1 \int x_2^2 |\psi_b(x_2)|^2 \, dx_2 = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 \, dx_1 \int x_2 |\psi_b(x_2)|^2 \, dx_2 = \langle x \rangle_a \langle x \rangle_b$$

$$\langle (x_1 - x_2)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$$

The main "punchline" is that the distance we just deduced actually **changes** for the symmetric and antisymmetric combinations (see proof on page 204):

bosons | bosons

$$\langle (x_1 - x_2)^2 \rangle_{\pm} = \underbrace{\langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b}_{\text{distinguishable}} \mp 2|\langle x \rangle_{ab}|^2$$

fermions | fermions

distinguishable

So even non-interacting particles (no Coulomb repulsion at all) suffer an effective force: bosons attract and fermions repel, in the sense that bosons are closer than classical particles and fermions are further apart.

This is a quantum effect, no classical analog!

Sec. 5.1.3. Now all together: the space dependence and the spin (still in 1D for simplicity)

$$\psi_S(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)]$$

$$\psi_{AS}(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)]$$

$$\chi_S(\mathbf{S}_1,\mathbf{S}_2) = \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \qquad \chi_{AS}(\mathbf{S}_1,\mathbf{S}_2) = \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$$

For 2 electrons (i.e. fermions) in two different states:

$\Psi = \psi_S(\mathbf{r}_1,\mathbf{r}_2)\,\chi_{AS}(\mathbf{S}_1,\mathbf{S}_2)$ OK !

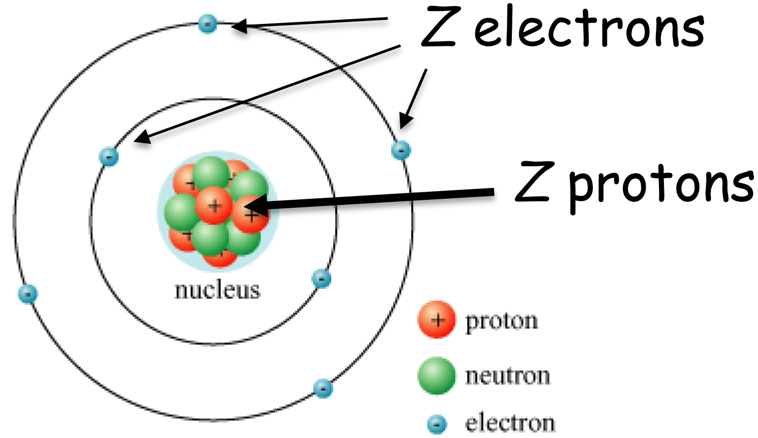
$\Psi = \psi_{AS}(\mathbf{r}_1,\mathbf{r}_2)\,\chi_S(\mathbf{S}_1,\mathbf{S}_2)$ OK !

$\Psi = \psi_S(\mathbf{r}_1,\mathbf{r}_2)\,\chi_S(\mathbf{S}_1,\mathbf{S}_2)$ NO

$\Psi = \psi_{AS}(\mathbf{r}_1,\mathbf{r}_2)\,\chi_{AS}(\mathbf{S}_1,\mathbf{S}_2)$ NO

If $a=b$, then only $\Psi = \psi_S(\mathbf{r}_1,\mathbf{r}_2)\,\chi_{AS}(\mathbf{S}_1,\mathbf{S}_2)$ possible (Pauli !!).

5.2 Atoms



The Hamiltonian is:

$$H = \sum_{j=1}^{Z} \left\{ -\frac{\hbar^2}{2m}\nabla_j^2 - \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Ze^2}{r_j} \right\} + \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\right)\sum_{j\neq k}^{Z} \frac{e^2}{|\mathbf{r}_j - \mathbf{r}_k|}$$

one-body e-p attraction
"the easy part"

e-e repulsion
"the difficult part"

Solving exactly is impossible ☹

$$H\ \psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_Z)\chi(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_Z) =$$

$$= E\ \psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_Z)\chi(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_Z)$$

No **r** dependence here