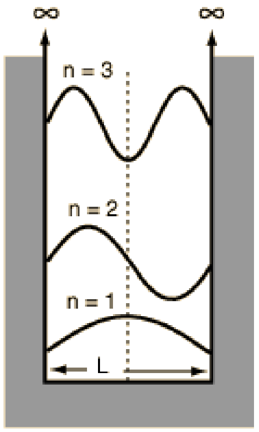
Example 5.1: Consider two particles without spin in the 1D infinite square well

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), \qquad E_n = n^2 K$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$



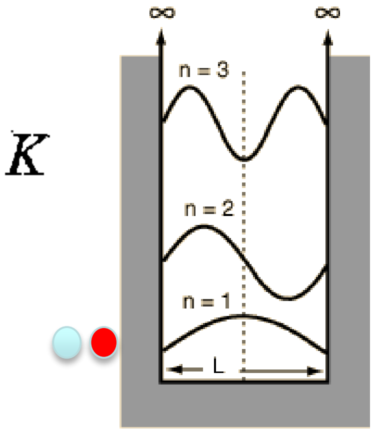
x = 0 at left wall of box.

L=a

(1) If particles were distinguishable, then

$$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2), \qquad E_{n_1 n_2} = (n_1^2 + n_2^2)K$$

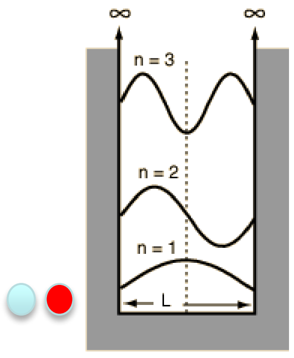
For ground state $n_1 = n_2 = 1$:



x = 0 at left wall of box.

Ground state n_1 = n_2 =1:

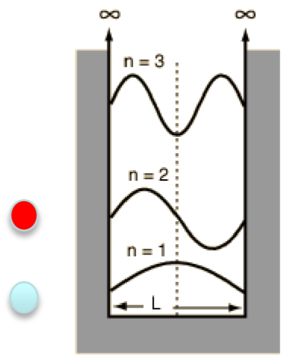
$$\boxed{\psi_{11} = \frac{2}{a} \sin(\pi x_1/a) \sin(\pi x_2/a),} \qquad E_{11} = 2K$$



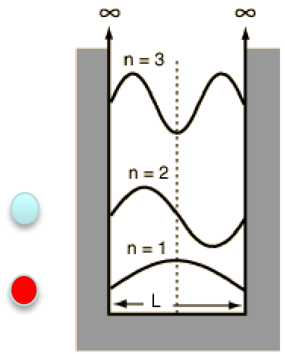
x = 0 at left wall of box.

First excited state n_1 = 1, n_2 = 2 or n_1 = 2, n_2 = 1

$$\boxed{\begin{aligned} \psi_{12} &= \frac{2}{a} \sin(\pi x_1/a) \sin(2\pi x_2/a), \\ \psi_{21} &= \frac{2}{a} \sin(2\pi x_1/a) \sin(\pi x_2/a), \end{aligned}} \qquad \begin{aligned} E_{12} &= 5K \\ E_{21} &= 5K \end{aligned}$$



x = 0 at left wall of box.

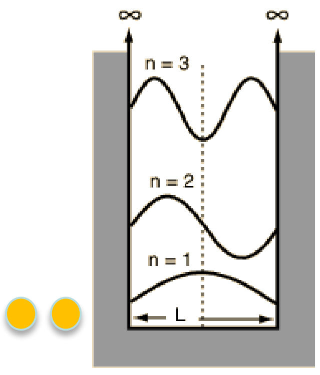


x = 0 at left wall of box.

Degeneracy = 2

(2) If particles are bosons:

Ground state (E=2K) is $n_1 = n_2 = 1$ for both:



x = 0 at left wall of box.

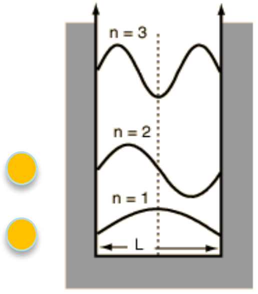
$$\boxed{\psi_{11} = \frac{2}{a}\sin(\pi x_1/a)\sin(\pi x_2/a),} \qquad E_{11} = 2K$$

$$\hat{P}\,\psi(x_1,x_2) = \psi(x_2,x_1) = +\psi(x_1,x_2)$$

First excited state (E=5K) is now nondegenerate

$$\boxed{\psi^{\text{excited}}(x_1,x_2) = \frac{\sqrt{2}}{a}\left[\sin(\pi x_1/a)\sin(2\pi x_2/a) + \sin(2\pi x_1/a)\sin(\pi x_2/a)\right]}$$

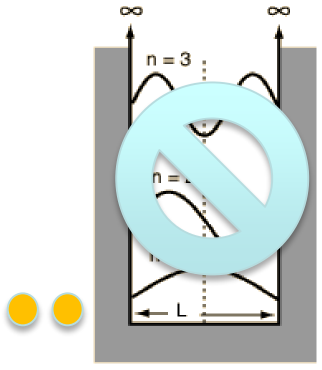
$$\hat{P}\,\psi^{\text{excited}}(x_1,x_2) = \psi^{\text{excited}}(x_2,x_1) = +\psi^{\text{excited}}(x_1,x_2)$$



x = 0 at left wall of box.

(3) If particles are fermions:

Because temporarily I am forgetting about the spin, then I cannot place 2 particles in $n_1 = 1$.

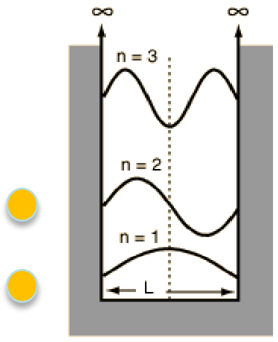


x = 0 at left wall of box.

Then, the true ground state (E=5K) becomes:

$$\boxed{\psi(x_1, x_2) = \frac{\sqrt{2}}{a}\left[\sin(\pi x_1/a)\sin(2\pi x_2/a) - \sin(2\pi x_1/a)\sin(\pi x_2/a)\right]}$$

$$\hat{P}\,\psi(x_1, x_2) = \psi(x_2, x_1) = -\psi(x_1, x_2)$$



x = 0 at left wall of box.

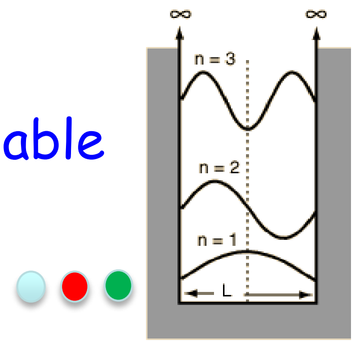
In summary (if spin is not included):

energy	distinguishable	bosons	fermions
$E=2K$	1 state	1 state	0 state
$E=5K$	2 states	1 state	1 state
etc			

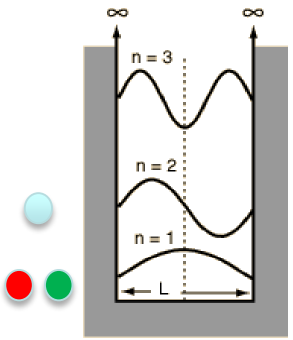
These differences have profound implications for many-body physics (condensed matter, nuclear, ...) and for their thermodynamics.

Three particles? (spin still not included)

distinguishable



x = 0 at left wall of box.



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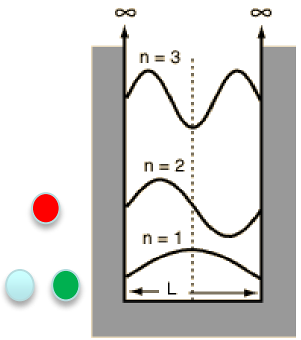
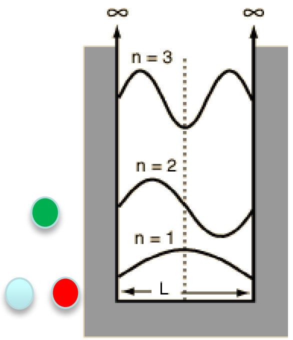
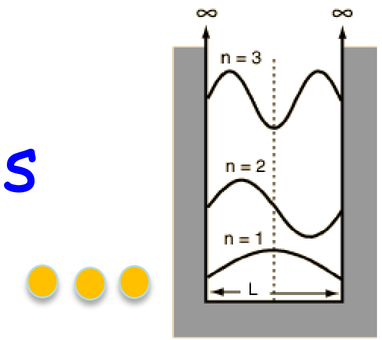


x = 0 at left wall of box.

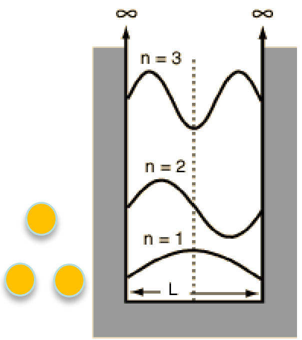


x = 0 at left wall of box.

bosons

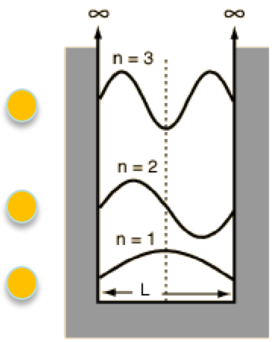


x = 0 at left wall of box.

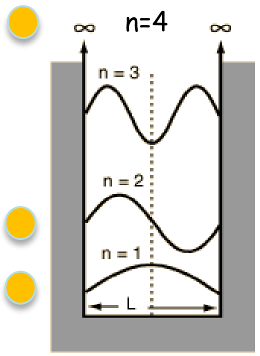


x = 0 at left wall of box.

fermions



x = 0 at left wall of box.



x = 0 at left wall of box.

5.1.2 Exchange Forces

This is an “effective force”, not a real force.

It is another of the consequences of having symmetric or antisymmetric combinations.
Consider 1D and two different states *a* and *b*.

distinguishable

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2)$$

bosons

$$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)]$$

fermions

$$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)]$$

Consider the typical "distance" between particles, via the expectation value:

$$\langle \underbrace{(x_1 - x_2)}_{\text{Distance in 1D, as example}}{}^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

distinguishable

$$\langle x_1^2 \rangle = \int x_1^2 |\psi_a(x_1)|^2 \, dx_1 \int |\psi_b(x_2)|^2 \, dx_2 = \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \int |\psi_a(x_1)|^2 \, dx_1 \int x_2^2 |\psi_b(x_2)|^2 \, dx_2 = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 \, dx_1 \int x_2 |\psi_b(x_2)|^2 \, dx_2 = \langle x \rangle_a \langle x \rangle_b$$

$$\boxed{\langle (x_1 - x_2)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b}$$

The main “punchline” is that the distance we just deduced actually **changes** for the symmetric and antisymmetric combinations (see proof on page 204):

$$\langle (x_1 - x_2)^2 \rangle_{\overset{\text{bosons}}{\underset{\text{fermions}}{\pm}}} = \underbrace{\langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b}_{\text{distinguishable}} \overset{\text{bosons}}{\underset{\text{fermions}}{\mp}} 2|\langle x\rangle_{ab}|^2$$

So even non-interacting particles (no Coulomb repulsion included) are under an effective force: bosons attract and fermions repel, in the sense that bosons are closer than classical particles and fermions are further apart.

This is a quantum effect; it has no classical analog.

Note: the expression used in the previous page for bosons and fermions contains a new term as compared with "distinguishable". The definition of the expression used there is the following.

$$|\langle x\rangle_{ab}|^2 \quad = \quad \langle x\rangle_{ab}\langle x\rangle_{ba} \quad =$$

$$\int x_1\psi_a(x_1)^*\psi_b(x_1)\,dx_1 \int x_2\psi_b(x_2)^*\psi_a(x_2)\,dx_2$$