4.4: Spin of electrons

As a warm up for P412 2021 we will quickly review the **last 2 lectures of 2020**, related to the spin of electrons.

A classical rigid body, like a planet, can have two kinds of angular momenta: (1) **L**, the orbital one associated with the center of mass, like Earth around the sun, and (2) **S**, the spin, like Earth rotating daily about an axis.

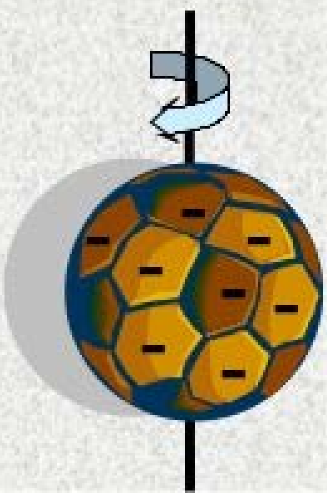
In quantum mechanics we discussed in P411 the orbital component **L** (related with the electron around the nucleus).

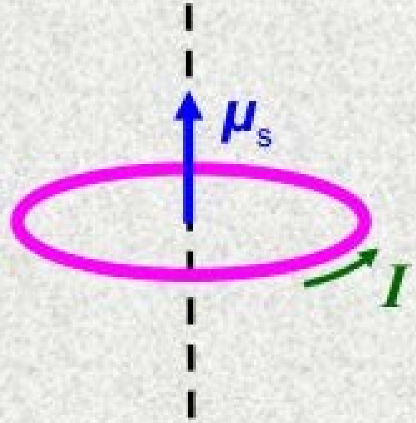
In QM, we also have a spin **S** for the electron but … the electron to the best of our accuracy is a POINT, thus cannot rotate.

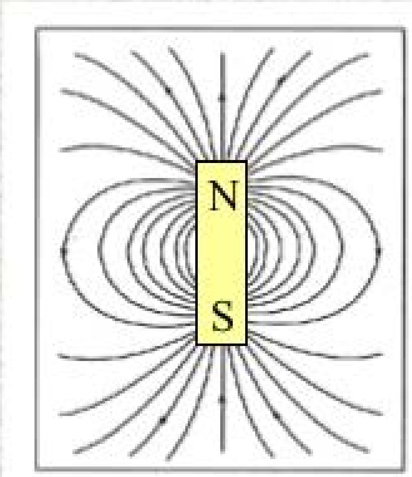
Spin Magnetic Dipole Moment

Just as electrons have the intrinsic properties of mass and charge, they have an intrinsic property called spin. This means that electrons, by their very nature, possess these three attributes. You're already comfortable with the notions of charge and mass. To understand spin it will be helpful to think of an electron as a rotating sphere or planet. However, this is no more than a helpful visual tool.

Imagine an electron as a soccer ball smeared with negative charge rotating about an axis. By the right hand rule, the angular momentum of the ball due to its rotation points down. But since its charge is negative, the spinning ball is like a little current loop flowing in the direction opposite its rotation, and the ball becomes an electromagnet with the N pole up. For an electron we would say its spin magnetic dipole moment vector, $\boldsymbol{\mu}_s$, points up. Because of its spin, an electron is like a little bar magnet.

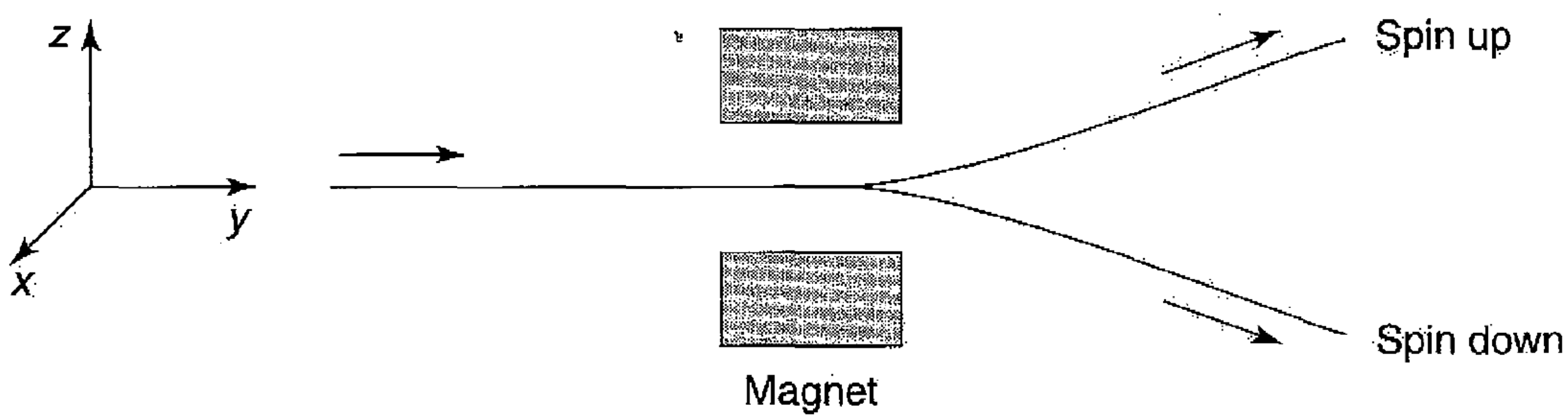


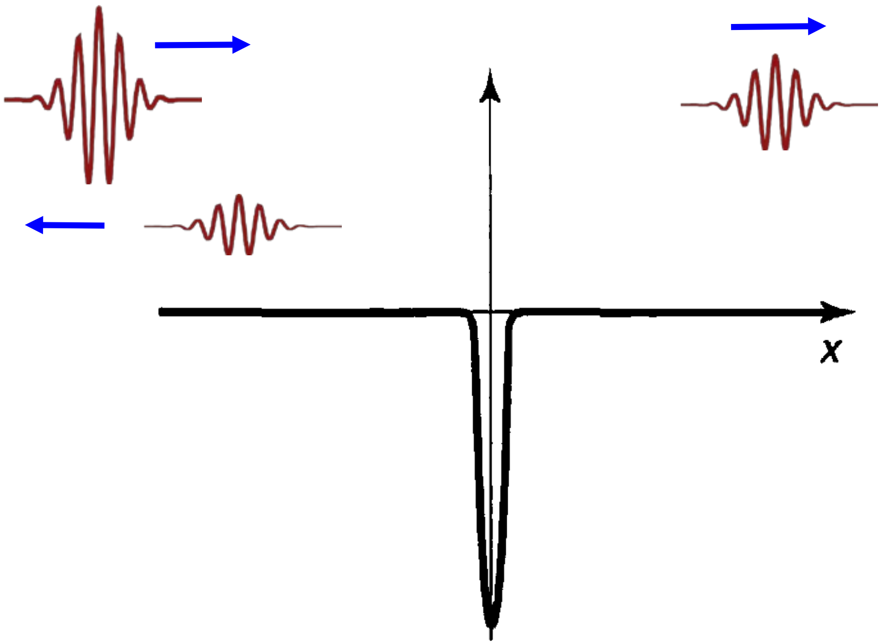




How do you know the electron has a spin? You introduce the particle in a magnetic field and use the formula $E = -\mu.\mathbf{B}$ (note μ and **S** are opposite for an e^-)

In the 1921 Stern-Gerlach experiment a beam of electrons was used (actually a beam of silver atoms which according to the electron counting should have 1e in the outer 5s l=0 level).





However, once we measure and find the particle at position x_0, then that "sigma" width is gone. The particle is exactly at x_0. At that moment what radius it has?

These wave packets have a "finite size" due to σ_x.

This wave-function finite size is related to the probability of finding the particle.

It seems that the radius is smaller than 10^-18 m according to experiments. Radius of nucleus is 10^-15 m. See URL discussing this topic in our web page.

Thus, it is a fact of Nature, that point-like particles, as an electron, carry an intrinsic spin angular momentum **S**.

Because the electron is a point, we cannot use the classical formula $\mathbf{S} = I\boldsymbol{\omega}$.

To describe the intrinsic spin the math leads. It has to be "analogous" to that of **L**. Let us start with the commutators:

$$[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y$$

becomes …

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y$$

The eigenfunctions are more "abstract" ...

First, let us switch to the Ch. 3 notation using an abstract Hilbert space notation "$|\alpha>$" for states:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m$$

becomes

$$L^2 \,|l\, m_l> = \hbar^2\, l(l+1)\,|l\, m_l>; \quad L_z\,|l\, m_l> = \hbar\, m_l\,|l\, m_l>$$

For L^2 and L_z using $Y_l^m(\theta, \phi)$ or $|l\, m_l>$ is the SAME.

But for the intrinsic spin, the Ch. 3 notation is the ONLY way.

Because in P411 we arrived to the eigenvalues by **only** using the commutators, then we simply repeat the operation line by line and find:

$$S^2 \, |s\, m_s> = \hbar^2 s(s+1) \, |s\, m_s> ; \quad S_z \, |s\, m_s> = \hbar \, m_s \, |s\, m_s>$$

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots ; \quad m = -s, -s+1, \ldots, s-1, s$$

The spin of each type of particle is FIXED, not like the orbital angular momentum L in H atom that you can change by emission or absorption of energy.

However, the projection of the spin of an electron can be changed, for instance by a magnetic field, but **the magnitude is intrinsic and fixed.**

4.4.1: Spin ½ *(electrons, quarks)*

Use $S^2 |s\, m_s\rangle = \hbar^2 s(s+1) |s\, m_s\rangle$; $S_z |s\, m_s\rangle = \hbar\, m_s |s\, m_s\rangle$

Specialize for s=1/2. Then, there are only two states, which in abstract form are:

$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$ and $|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

We call them spin "up", or ↑, and spin "down", or ↓ .

There is another, still abstract, way to represent spins up and down. It is using so-called "**spinors**"

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can combine the "up" and "down" linearly at will.
So the spin could point, e.g., "sideways".

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

If we use spinors for the states, then what do we use for the **operators** such as L^2 ? Certainly we cannot use derivatives of angles as for spherical harmonics.

From the two equations …

$$S^2 \mid \tfrac{1}{2} \;\; \tfrac{1}{2} > = \hbar^2 \, \tfrac{1}{2} \left(\tfrac{1}{2} + 1\right) \mid \tfrac{1}{2} \;\; \tfrac{1}{2} >$$

$$S^2 \mid \tfrac{1}{2} \; -\tfrac{1}{2} > = \hbar^2 \, \tfrac{1}{2} \left(\tfrac{1}{2} + 1\right) \mid \tfrac{1}{2} \; -\tfrac{1}{2} >$$

same

… it can be deduced (see book, easy) that:

$$\mathbf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From the other two equations …

$$S_z \,|\, \tfrac{1}{2} \;\; \tfrac{1}{2}\rangle = \tfrac{1}{2}\hbar \,|\, \tfrac{1}{2} \;\; \tfrac{1}{2}\rangle$$

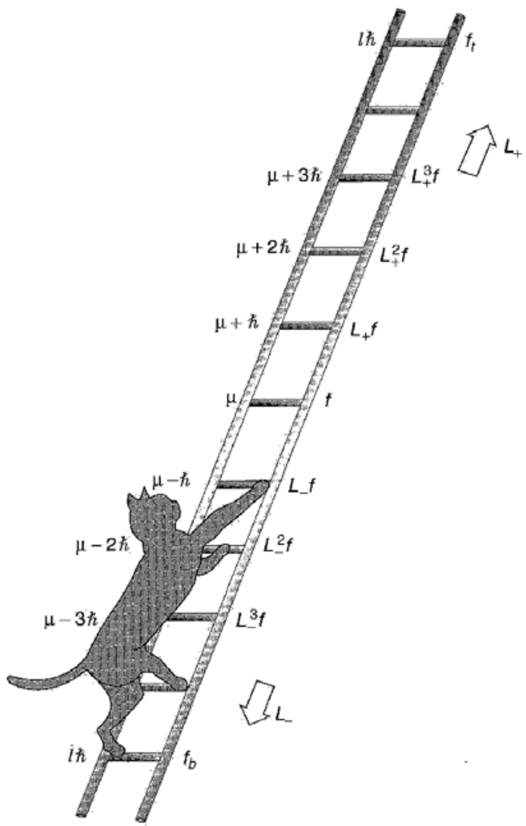
$$S_z \,|\, \tfrac{1}{2} \; -\tfrac{1}{2}\rangle = -\tfrac{1}{2}\hbar \,|\, \tfrac{1}{2} \; -\tfrac{1}{2}\rangle$$

… it can deduced (see book, easy) that:

$$\mathbf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

There has to be also an analog of the raising and lowering operators:

$$L_{\pm} = L_x \pm i L_y$$



$$\mathbf{S}_+ \chi_- = \hbar \chi_+$$

$$\mathbf{S}_- \chi_+ = \hbar \chi_-$$

$$\mathbf{S}_+ \chi_+ = \mathbf{S}_- \chi_- = 0$$

$$\mathbf{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Example:

$$\hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Recalling $S_\pm = S_x \pm i S_y$ then (see book, page 168):

$$S_x = (1/2)(S_+ + S_-) \qquad S_y = (1/2i)(S_+ - S_-)$$

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Dropping the ħ/2 factor defines
the famous **Pauli matrices**:

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Returning to the general combination:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_{+} + b\chi_{-}$$

It has to be normalized like any other state i.e.

$$|a|^2 + |b|^2 = 1$$

$|a|^2$ is the probability of measuring spin up.

$|b|^2$ is the probability of measuring spin down.

All together now:

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix}$$

Set *nlm* replaced by $nlm_l s m_s$. Above is $nlm_l \frac{1}{2}\frac{1}{2}$
Now we need 5 quantum numbers to represent the state of one electron in a hydrogen atom.

Comment: The total magnetic moment **J** arises from the sum as vectors of **L** and **S**. This will be discussed in Ch. 5.

Returning, again!, to the general combination:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

What is the probability that the spin points say along the *positive x axis*?

To answer this question, first you have to diagonalize the 2x2 Pauli matrix "σ_x". Turns out the "eigenspinors" are:

$$\chi_+^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \left(\text{eigenvalue} + \frac{\hbar}{2}\right); \quad \chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \left(\text{eigenvalue} - \frac{\hbar}{2}\right)$$

Petite summary for your convenience:

For x Pauli matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$+\frac{\hbar}{2} \quad \text{Eigenvalues} \quad -\frac{\hbar}{2}$$

For z Pauli matrix:

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

$$+\frac{\hbar}{2} \quad \text{Eigenvalues} \quad -\frac{\hbar}{2}$$

The general combination ...

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

... can now be written as:

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}$$

$$(1/2)|a+b|^2$$

is the probability of measuring spin up along x.

$$(1/2)|a-b|^2$$

is the probability of measuring spin down along x.

The previous page was the last topic we covered in P411 … now we truly start P412.

Let us see how the formulas work in practice. Start with:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider example 4.2 book, page 169. The arbitrarily chosen spin state given here is

$$\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

Then, $a = (1+i)/\sqrt{6}$ and $b = 2/\sqrt{6}$

prob. of $+\hbar/2$ if S_z measured $= |(1+i)/\sqrt{6}|^2 = 1/3$

prob. of $-\hbar/2$ if S_z measured $= |2/\sqrt{6}|^2 = 2/3$

In the previous page we used the eigenspinors of the *z*-axis Pauli matrix:

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

Before we mentioned briefly the eigenspinors of the x Pauli matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Make sure you know how to confirm this last result!

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \longrightarrow \begin{vmatrix}-\lambda & \hbar/2\\ \hbar/2 & -\lambda\end{vmatrix} = 0 \longrightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

x Pauli matrix

Write determinant of S_x. λ is the eigenvalue.

Solve determinant; find eigenvalues

Finally find eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longrightarrow \begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\beta = \pm\alpha$$

... and finally normalize to 1.

Consider now the SAME spinor of example 4.2 but from the perspective of the eigenspinors of the x Pauli matrix. You can use any basis after all.

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a = (1+i)/\sqrt{6}$$
$$b = 2/\sqrt{6}$$

$+\hbar/2$: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (→ $\chi_+^{(x)}$)

$-\hbar/2$: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (→ $\chi_-^{(x)}$)

$\left|\frac{a+b}{\sqrt{2}}\right|^2$ = prob. of getting $+\hbar/2$ if S_x is measured $= (1/2)|(3+i)/\sqrt{6}|^2 = 5/6$

Until you confirm these results by yourself, you will NOT understand.