$$\left|\frac{a-b}{\sqrt{2}}\right|^2$$ = prob. of getting $-\hbar/2$ if $S_x$ is measured $= (1/2)|(-1+i)/\sqrt{6}|^2 = 1/6$

prob. of getting $+\hbar/2$ or $-\hbar/2$ if $S_x$ is measured has to be 1.

Indeed 5/6 + 1/6 =1.

What is the "expectation value" of $S_x$? I.e. what is $\langle S_x \rangle$?

$$\frac{5}{6}\left(+\frac{\hbar}{2}\right)+\frac{1}{6}\left(-\frac{\hbar}{2}\right)=\frac{\hbar}{3}$$

Petite challenge for yourself. What is $\langle S_z \rangle$? How about $\langle S_y \rangle$? I want this to be an opportunity for students to interact.

Alternatively, you get the same $\langle S_x \rangle = \hbar/3$ as follows:

Check last step! $(e,f)\begin{pmatrix} c \\ d \end{pmatrix} = e.c+f.d$

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\sigma_x$ Pauli matrix

$$\frac{\hbar}{2}\begin{pmatrix} 2/\sqrt{6} \\ (1+i)/\sqrt{6} \end{pmatrix}$$

$$\langle S_x \rangle = \chi^{\dagger}\mathbf{S}_x\chi = \left(\frac{(1-i)}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}\begin{pmatrix} (1+i)/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} = \frac{\hbar}{3}$$

horizontal spinor with each component conjugated !!

vertical spinor as originally given, normalized to 1

Until you confirm these results by yourself, you will NOT be sure you REALLY know how to deal with 2x2 matrices.

**Side comment:** The spin appears naturally from the Dirac equation that links quantum mechanics and relativity.

Adapted from Wikipedia on Dirac equation:

*(The Dirac eq.) provided a theoretical justification for the introduction of several component wave functions in Pauli's phenomenological theory of spin (note: Pauli theory is what we are doing); the wave functions in the Dirac theory are spinors of four complex numbers, two of which resemble the Pauli spinors in the non-relativistic limit, in contrast to the Schrödinger equation which described wave functions with only one complex value.*

# 4.4.2 Electron in a magnetic field (page 172)

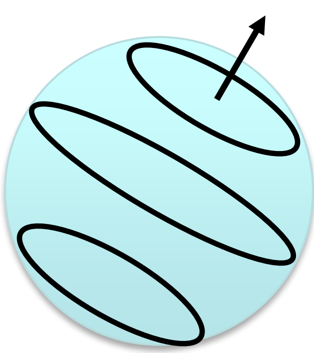
When an electron is at rest inside a uniform magnetic field $\mathbf{B}$, the Hamiltonian can be shown to be:

$$H = -\boldsymbol{\mu}\cdot\mathbf{B} = -\gamma\mathbf{B}\cdot\mathbf{S}$$

$\boldsymbol{\mu} = \gamma\,\mathbf{S}$

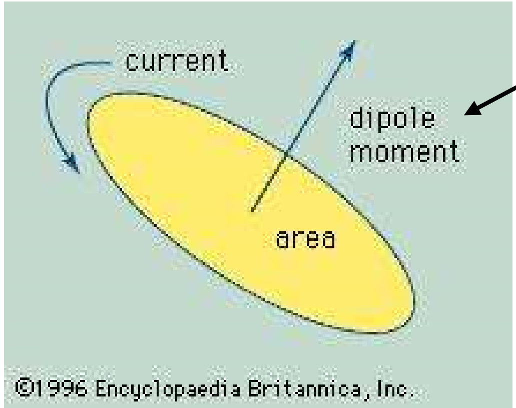
$\gamma$ = "gyromagnetic ratio"
=-|e|/m for electrons
(factor 2 difference with classical E&M calculation due to relativity)

Repeating: A spinning charged object is made of little loops of current.



total charge q
total mass m

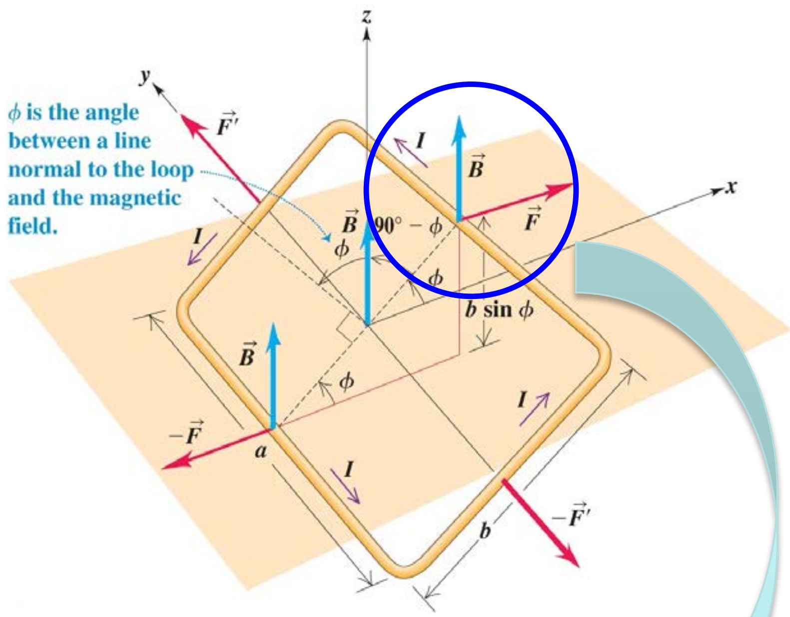
Each little loop generates a little magnetic field. They add up to a net dipole magnetic moment.




This is "μ".

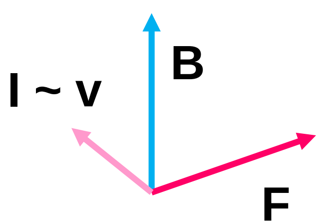
If q<0, then current and dipole invert; μ and S are **antiparallel**

**Not in book:** When a little loop of current is introduced in a strong external magnetic field, the Lorentz force on the moving charges generates a torque.




The torque tries to align the dipole moment with **B**, and the energy is found to be in E&M

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$




Lorentz force
F ~ v x B

Suppose for simplicity that the field points along the *z*-axis i.e. $\mathbf{B} = (0,0,1)B_0$. Then,

$$\mathsf{H} = -\gamma B_0 \mathsf{S}_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Change of notation to bold only to emphasize 2x2 character. Optional but recommended!

Eigenfunctions, or eigenspinors, are:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{H}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(\gamma B_0 \hbar)/2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For spin up, this is energy eigenvalue $E_+$

For spin down, the energy $E_-$ changes sign.

The Hamiltonian is time independent because the magnetic field is constant. Actually, all Hamiltonians we studied in P411 were time independent.

If H is time independent, then the general solution is a linear combination of stationary states. In the square well potential there were infinite number of states (Ch2 P411)

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

Focusing only on S=1/2, all much easier. Just two states! The most general spinor is (page 173):

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix}$$

The values of "$a$" and "$b$" are fixed by initial condition at $t=0$, as we did in Ch. 2 for the coefficients $c_n$.

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

We also need to normalize: $|a|^2 + |b|^2 = 1$

When we have 2 unknowns linked as in the normalization condition, we can parametrize with an angle as $a = \cos(\alpha/2)$ $b = \sin(\alpha/2)$

because

$$|a|^2 + |b|^2 = \cos^2(\alpha/2) + \sin^2(\alpha/2) = 1$$

The factor ½ for the angle is only for future convenience in next pages.

Then, we arrive to a simple formula:

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

Physical meaning of angle $\alpha$? Let us repeat what we did before in the example $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, but now for the spinor in a magnetic field.

$$\langle S_x \rangle = \chi(t)^\dagger \mathbf{S}_x \chi(t) = \left(\cos(\alpha/2)e^{-i\gamma B_0 t/2},\ \sin(\alpha/2)e^{i\gamma B_0 t/2}\right)$$

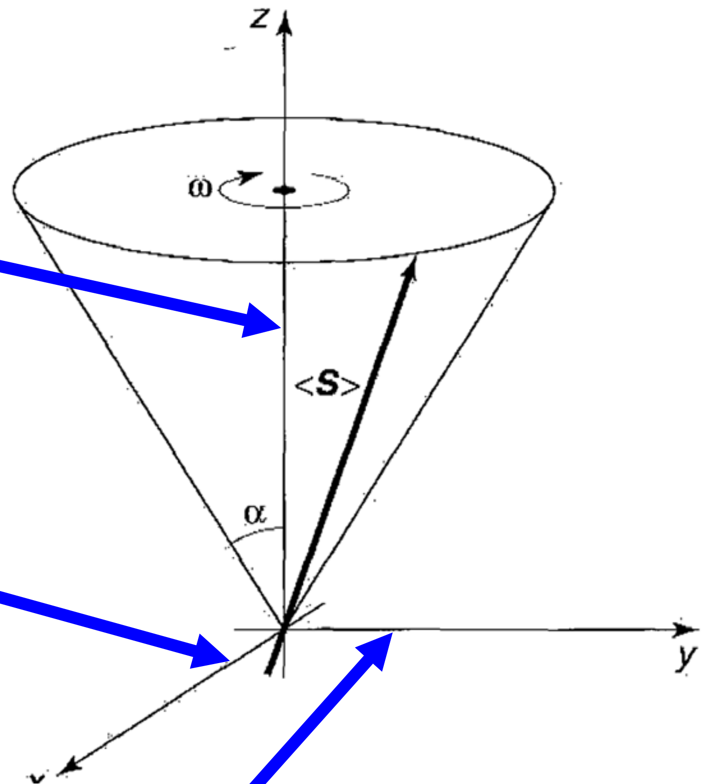
$$\times \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t).$$

Make sure you verify this calculation! Same for $\langle S_y \rangle$ and $\langle S_z \rangle$, page 173.

Repeating for the other components:

$$\langle S_z \rangle = \chi(t)^\dagger \mathbf{S}_z \chi(t) = \frac{\hbar}{2}\cos\alpha$$




$$\langle S_x \rangle = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$$

$$\langle S_y \rangle = \chi(t)^\dagger \mathbf{S}_y \chi(t) = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$$

This is the same as a classical dipole oscillating in a magnetic field. It is "precessing" …