Not in book:

Why the electron precesses instead of simply aligning with the magnetic field to minimize energy?

REASON: the energy is conserved -- the electron cannot change its energy -- because the Hamiltonian is time independent!

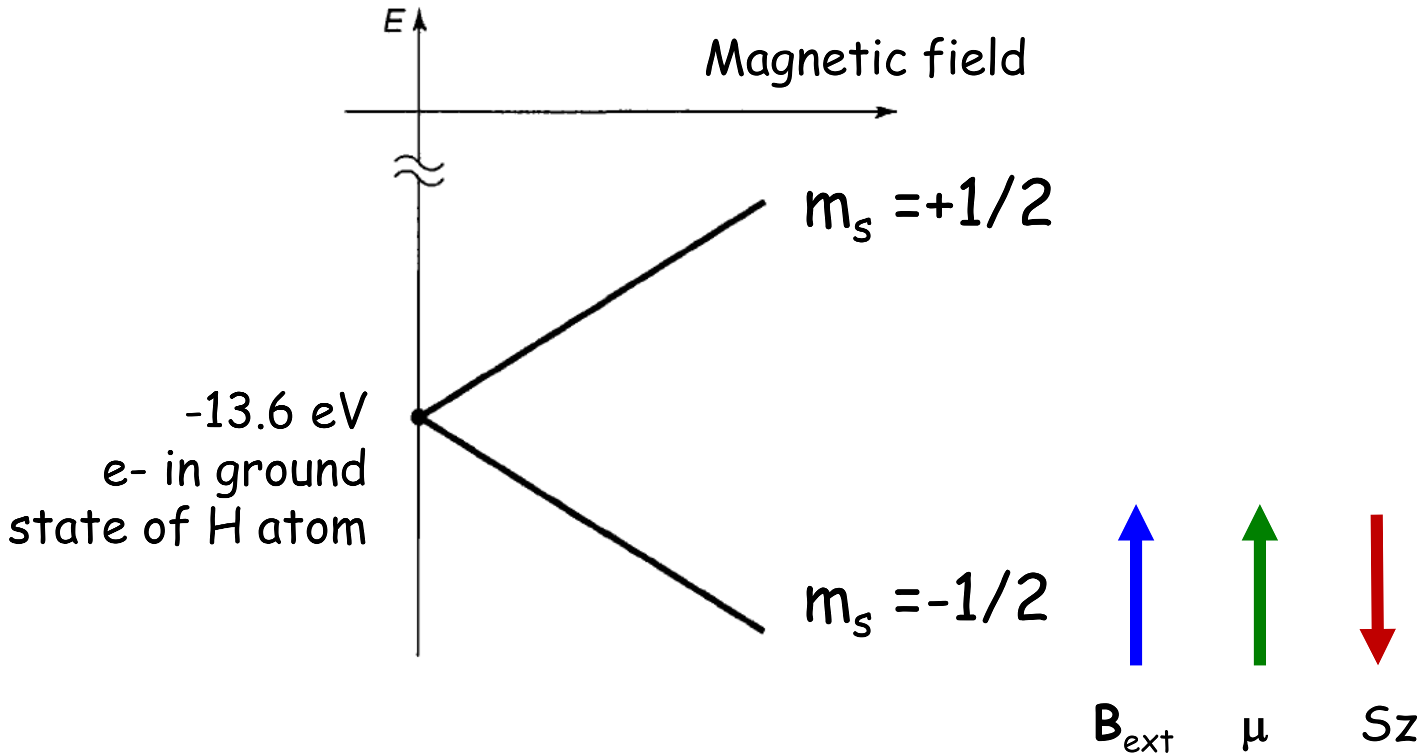
Consider $\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$

$$\langle \mathbf{H} \rangle = \chi(t)^{\dagger}\, \mathbf{H}\, \chi(t) = E_{+} \cos^2(\alpha/2) + E_{-} \sin^2(\alpha/2)$$

where $E_{+} = -(\gamma B_0 \hbar)/2$
$E_{-} = +(\gamma B_0 \hbar)/2$

Time independent! The energy does not change in a t-independent Hamiltonian

Anticipating what is coming in Ch. 7:



$$H = (|e|/m)\, \mathbf{B}_{ext} \cdot \mathbf{S}$$

$(\gamma < 0)$

4.4.3 Addition of angular momenta (page 176)

First time we go beyond one electron!
Suppose we have TWO particles, each with spin ½.

We focus on the spin, not the actual relative motion. Also could be 2 electrons or 1 electron and 1 proton.

Because each spin can be up or down, overall, regardless of real distance between electrons, we have 4 cases:

$$\uparrow\uparrow, \quad \uparrow\downarrow, \quad \downarrow\uparrow, \quad \downarrow\downarrow$$

We want to know: what is the **total** spin? Natural answers are 1=1/2 +1/2 or 0=1/2-1/2. But why not other numbers? Classically, any number between 1 and 0 is good.

Note: We will NOT deal with the **Clebsch-Gordan** coefficients but you will have all the foundations to handle that page of the book.

means: operators only of particle 1

The total spin operator is: $\mathbf{S} \equiv \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$

The z-component of the total spin operator is:

$$S_z = S_z^{(1)} + S_z^{(2)}$$

$$\uparrow\uparrow, \ \uparrow\downarrow, \ \downarrow\uparrow, \ \downarrow\downarrow$$

$$S_z \chi_1 \chi_2 = (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2 = (S_z^{(1)} \chi_1) \chi_2 + \chi_1 (S_z^{(2)} \chi_2)$$

$$= (\hbar m_1 \chi_1) \chi_2 + \chi_1 (\hbar m_2 \chi_2) = \hbar \underbrace{(m_1 + m_2)}_{m} \chi_1 \chi_2$$

E.g. for particle 1, this can be up or down along the *z* axis

$$\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$$

$$S_z \chi_1 \chi_2 = \hbar\, m\, \chi_1 \chi_2$$

$$\begin{aligned} \uparrow\uparrow&: \quad m = 1; \\ \uparrow\downarrow&: \quad m = 0; \\ \downarrow\uparrow&: \quad m = 0; \\ \downarrow\downarrow&: \quad m = -1. \end{aligned}$$

What does this mean? There are 2 combinations with m=0? Use "lowering operators" (or "raising operators") to clarify and find special combinations.

$$S_- = S_-^{(1)} + S_-^{(2)}$$

$$\begin{aligned} S_-(\uparrow\uparrow) &= (S_-^{(1)}\uparrow)\uparrow + \uparrow(S_-^{(2)}\uparrow) \\ &= (\hbar\downarrow)\uparrow + \uparrow(\hbar\downarrow) = \hbar(\downarrow\uparrow + \uparrow\downarrow) \end{aligned}$$

This suggests the sum up down + down up "emerges" as special.

Not in book (but probably in HW):

$$S_-(\downarrow\uparrow + \uparrow\downarrow) = (S_-^{(1)} + S_-^{(2)})(\downarrow\uparrow + \uparrow\downarrow) =$$

$$= \underbrace{(S_-^{(1)}\downarrow)}_{=0}\uparrow + \underbrace{(S_-^{(1)}\uparrow)}_{(\hbar\downarrow)}\downarrow + \downarrow\underbrace{(S_-^{(2)}\uparrow)}_{(\hbar\downarrow)} + \uparrow\underbrace{(S_-^{(2)}\downarrow)}_{=0} =$$

$$= 2\hbar\downarrow\downarrow$$

This means that $\uparrow\uparrow$, $(\downarrow\uparrow + \uparrow\downarrow)$, and $\downarrow\downarrow$ are linked forming a set called the triplet (see (4.175) book, page 177)

This takes care of 3 states. How about number 4?

Make sure you understand this page and next, step by step.

Not in detail in book (probably in HW you will repeat):

$$S_-(\downarrow\uparrow - \uparrow\downarrow) = (S_-^{(1)} + S_-^{(2)})(\downarrow\uparrow - \uparrow\downarrow) =$$

$$= (\underbrace{S_-^{(1)}\downarrow}_{=0})\uparrow - (\underbrace{S_-^{(1)}\uparrow}_{(\hbar\downarrow)})\downarrow + \downarrow(\underbrace{S_-^{(2)}\uparrow}_{(\hbar\downarrow)}) - \uparrow(\underbrace{S_-^{(2)}\downarrow}_{=0}) =$$

$$= 0.$$

If you apply S_+ it gives zero as well. This means that $(\uparrow\downarrow - \downarrow\uparrow)$ is **alone** forming a singlet.

$$\left\{\begin{array}{ll} |1\,1\rangle & = \uparrow\uparrow \\ |1\,0\rangle & = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,-1\rangle & = \downarrow\downarrow \end{array}\right\} \quad s = 1$$

(triplet: 3 discrete projections)

$$\left\{|0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right\} \quad s = 0$$

(singlet: only 1 projection)

$|s\ m\rangle$

s denotes the total spin and *m* denotes the *z*-axis projection of the total spin.

What we called the "triplet" has three states, but is it truly a state of total spin 1?

Consider the **total** spin operator:

The complicated part

$$S^2 = (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) \cdot (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$$

Example for state "up down":

$$\begin{aligned}\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}(\uparrow\downarrow) &= (S_x^{(1)} \uparrow)(S_x^{(2)} \downarrow) + (S_y^{(1)} \uparrow)(S_y^{(2)} \downarrow) + (S_z^{(1)} \uparrow)(S_z^{(2)} \downarrow) \\ &= \left(\frac{\hbar}{2}\downarrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(\frac{i\hbar}{2}\downarrow\right)\left(\frac{-i\hbar}{2}\uparrow\right) + \left(\frac{\hbar}{2}\uparrow\right)\left(\frac{-\hbar}{2}\downarrow\right) \\ &= \frac{\hbar^2}{4}(2\downarrow\uparrow - \uparrow\downarrow)\end{aligned}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0 \\ i\end{pmatrix} = \frac{\hbar}{2} i \begin{pmatrix}0 \\ 1\end{pmatrix}$$

Similarly: $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} (\downarrow\uparrow) = \frac{\hbar^2}{4}(2\uparrow\downarrow - \downarrow\uparrow)$

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} |1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2\downarrow\uparrow - \uparrow\downarrow + 2\uparrow\downarrow - \downarrow\uparrow) = \frac{\hbar^2}{4}|1\,0\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$S^2|1\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|1\,0\rangle = 2\hbar^2|1\,0\rangle$$

Total spin squared

$(S^{(1)})^2$ $(S^{(2)})^2$ $2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$

S(S+1) = 1 (1+1) =2
for total spin

s(s+1) = 1/2 (1/2 + 1) =3/4
for individual spins

Same story with "singlet" (left as exercise):

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} |0\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2\downarrow\uparrow - \uparrow\downarrow -2\uparrow\downarrow + \downarrow\uparrow) = -\frac{3\hbar^2}{4}|0\,0\rangle$$

Just a sign difference with previous page!

$$S^2|0\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - 2\frac{3\hbar^2}{4}\right)|0\,0\rangle = 0$$

S(S+1) = 0 (0+1) =0 for total spin

Also left as exercise, application of total spin over "up up" $|1\,1\rangle$ and "down down" $|1\,-1\rangle$.