Welcome back to QM part 2 (P412)

Just a few comments before starting:

(a) In my group web page, under Teaching, you will find the new P412 Quantum Mechanics, Spring 2023

(b) We have a new grader
(PhD Grad Student **Seunghoon Song**,
email *ssong17@vols.utk.edu*).

Find his mailbox using the photos in the syllabus by focusing on the red frames (room 401B). Search for "SONG". Fourth from top, on the right.

Tests will be: **Test 1 on Feb. 23, Test 2 on April 4**.
The "final" is fixed by the University Academic Calendar, thus **Test3 will be on May 15**.

Thanks for the very generous comments in TNVoice.

One suggestion was to *provide an example on "how to find your grade exactly"* and now a complete example is provided in the P412 syllabus.

4.4.2 Electron in a magnetic field (page 172)

When an electron at rest is spinning in an uniform magnetic field $\mathbf{B}$, the Hamiltonian can be shown to be:

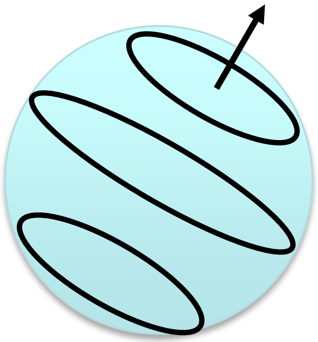
$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma \mathbf{B} \cdot \mathbf{S}$$

$\boldsymbol{\mu} = \gamma\, \mathbf{S}$

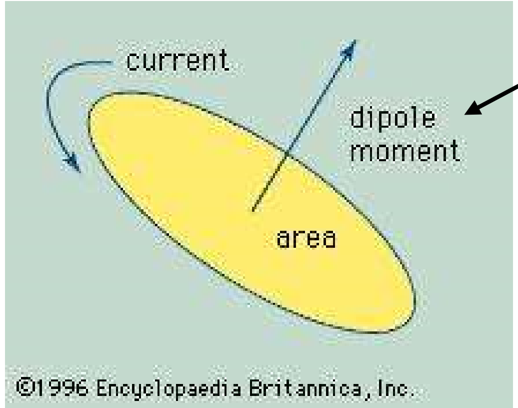
γ = "gyromagnetic ratio"
=-|e|/m for electrons
(factor 2 difference with classical E&M calculation due to relativity)

Intuition: A spinning charged object is made of little loops of current.

Each little loop generates a little magnetic field. They add up to a net dipole magnetic moment.



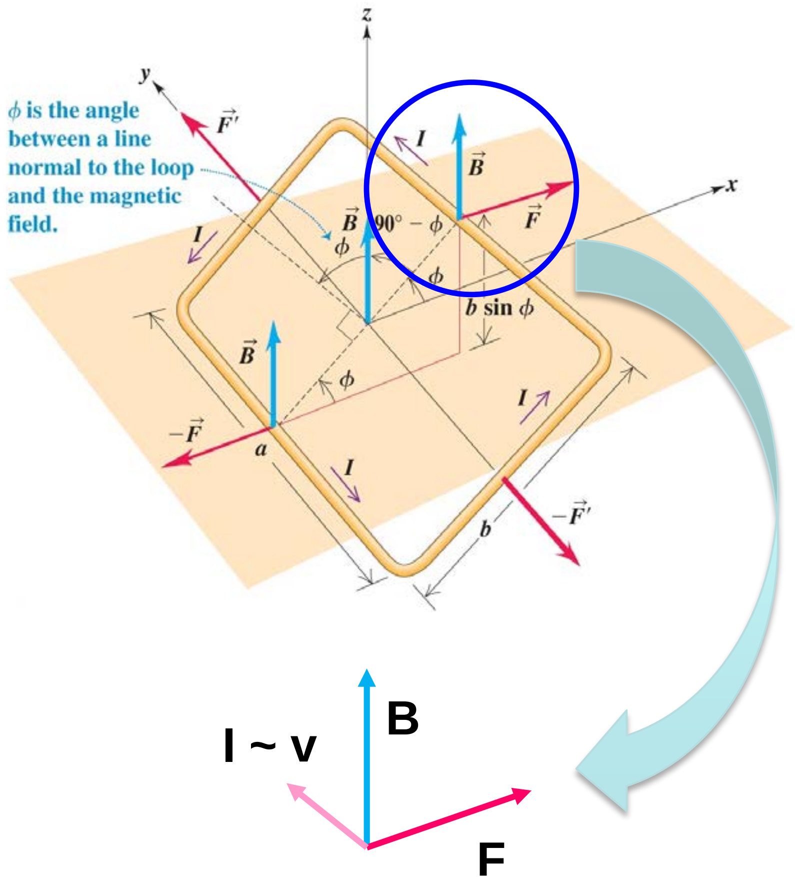
total charge q
total mass m



This is vector "μ".

If q<0, then current and dipole invert;
μ and S are antiparallel

Not in book: When a little loop of current is introduced in a strong external magnetic field, the Lorentz force on the moving charges generates a torque.



Lorentz force
F ~ v x B

The torque tries to align the dipole moment (perpendicular to the loop) with **B**, and in the E&M class the energy is found to be

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

For simplicity, suppose that the field points along the *z*-axis i.e. $\boldsymbol{B} = (0,0,1)B_0$. Then,

$$\mathbf{H} = -\gamma B_0 \mathbf{S}_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notation changed to bold to emphasize quantum 2x2 character. Optional but recommended.

Eigenfunctions, or eigenspinors, are:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{H}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(\gamma B_0 \hbar)/2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For spin up, this is energy eigenvalue E_+

For spin down, the energy E_- changes sign.

The Hamiltonian is time independent because the magnetic field is constant. Actually, all Hamiltonians we studied in P411 were time independent.

For time-indep. Hamiltonians, the general solution is a linear combination of stationary states. In the square well potential, there were infinite number of states (Ch2 P411), thus we used:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

When focusing only on S=1/2, all much easier. Just two states! The most general spinor is (page 173):

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix}$$

The values of "a" and "b" are fixed by initial condition at t=0, as we did in Ch. 2 for the coefficients c_n.

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

We also need to normalize: $|a|^2 + |b|^2 = 1$

When we have 2 unknowns linked as in the normalization condition, we can parametrize with an angle as $a = \cos(\alpha/2)$ $\quad$ $b = \sin(\alpha/2)$

because

$$|a|^2 + |b|^2 = \cos^2(\alpha/2) + \sin^2(\alpha/2) = 1$$

The factor ½ for the angle is only for future convenience in next pages.

Then, we arrive to a simple formula:

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

Physical meaning of angle α? Let us repeat what we did in P411 to get expectations values for $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, but now for the spinor in a magnetic field, shown above.

$$\langle S_x \rangle = \chi(t)^\dagger \mathbf{S}_x \chi(t) = \left(\cos(\alpha/2)e^{-i\gamma B_0 t/2},\ \sin(\alpha/2)e^{i\gamma B_0 t/2}\right)$$

$$\times \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t).$$

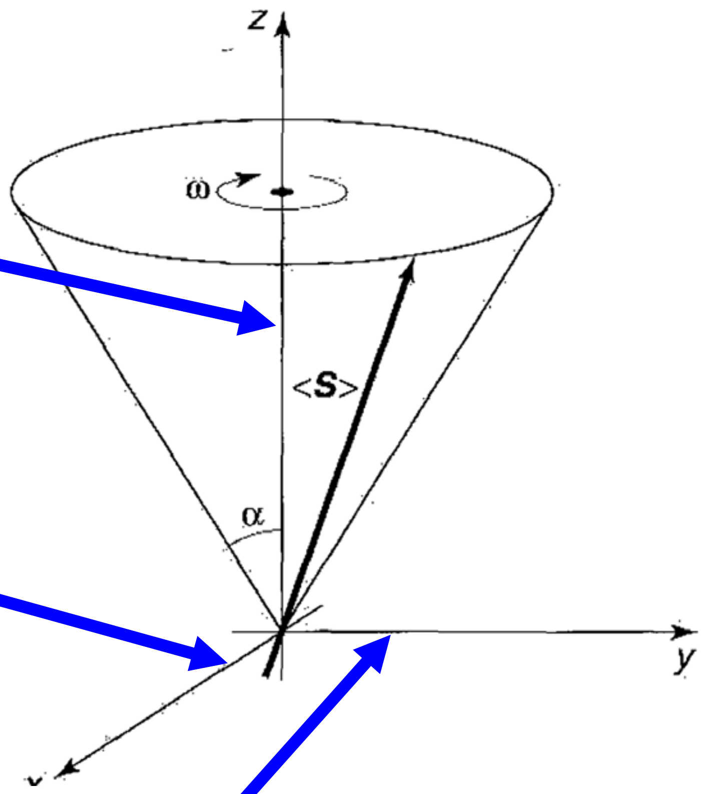
Make sure you verify this calculation! Same for $\langle S_y \rangle$ and $\langle S_z \rangle$, page 173.

Repeating for the other components:

$$\langle S_z \rangle = \chi(t)^\dagger \mathbf{S}_z \chi(t) = \frac{\hbar}{2}\cos\alpha$$

$$\langle S_x \rangle = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$$

$$\langle S_y \rangle = \chi(t)^\dagger \mathbf{S}_y \chi(t) = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$$



This is the same as a classical dipole oscillating in a magnetic field. It is "precessing" … Expectation values behave as the classical objects would.

Not in book:

Why the electron precesses instead of simply aligning with the magnetic field to minimize energy?

REASON: the energy is conserved -- the electron cannot change its energy -- because the Hamiltonian is time independent!

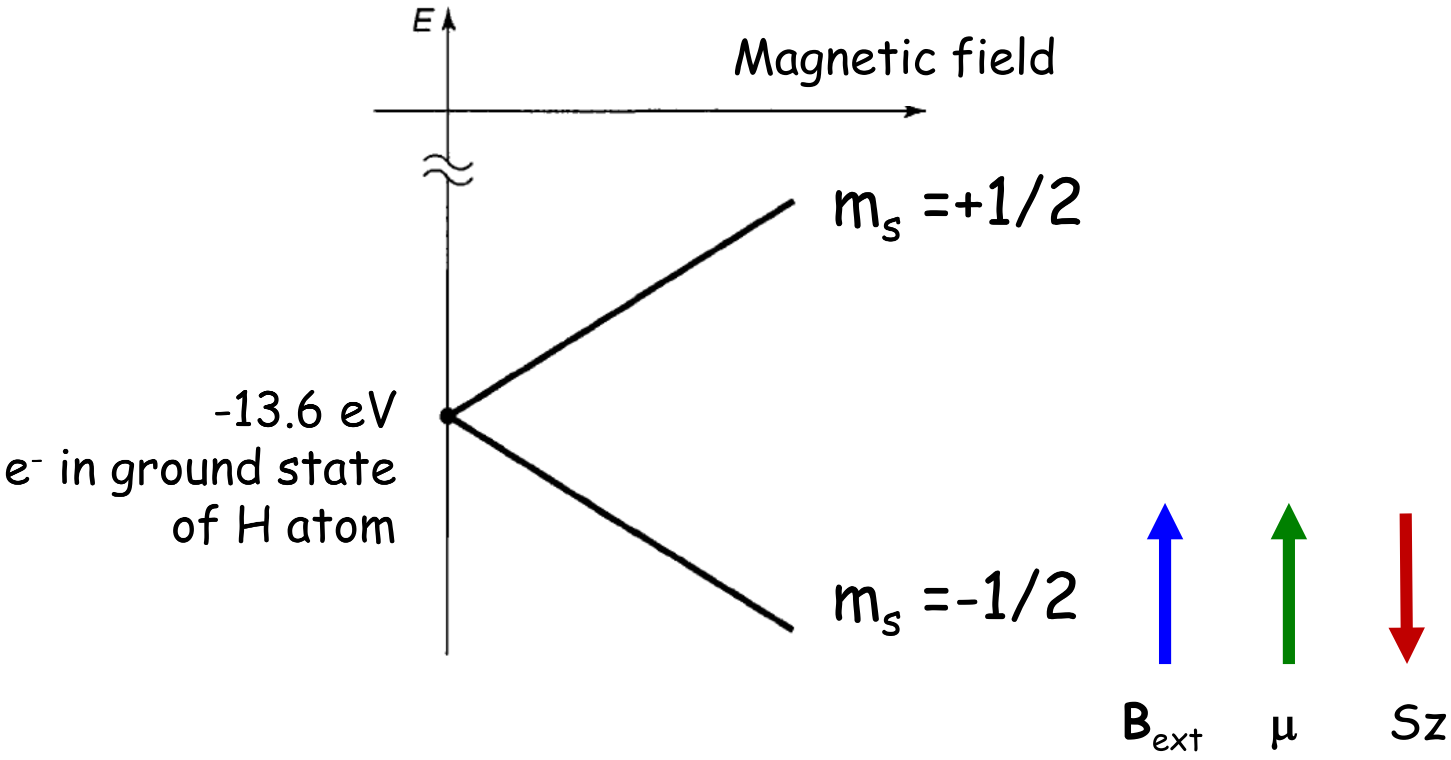
Consider $\chi(t) = \begin{pmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{pmatrix}$

$$\langle \mathbf{H} \rangle = \chi(t)^{\dagger}\, \mathbf{H}\, \chi(t) = E_{+} \cos^2(\alpha/2) + E_{-} \sin^2(\alpha/2)$$

where $E_{+} = -(\gamma B_0 \hbar)/2$
$E_{-} = +(\gamma B_0 \hbar)/2$

Time independent! The energy does **not** change in a *t*-independent Hamiltonian

Anticipating what is coming in Ch. 7.
This page will be explained again in the next lecture as well.



$H = (|e|/m)\,\mathbf{B}_{ext}.\mathbf{S}$

(because γ <0)