4.4.3 Addition of angular momenta (page 176)

Important: First time we go beyond one electron!
Suppose we have TWO particles, each with spin ½.

Here we focus on the spin, not the actual relative motion.
Could be 2 electrons or 1 electron and 1 proton …

Because each spin can be up or down, overall, *regardless of real distance between electrons*, we have 4 cases:

$$\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$$

We want to know: what is the **total** spin? Natural answers are 1=1/2 +1/2 or 0=1/2-1/2. But why not other numbers? Classically, any number between 1 and 0 is good.

This means operators of particle 1.
Now we have to label each particle.

The total spin operator is:

$$\mathbf{S} \equiv \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$$

The *z*-component of the total spin operator is:

$$S_z = S_z^{(1)} + S_z^{(2)}$$

$$\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$$

$$\begin{aligned} S_z\chi_1\chi_2 &= \left(S_z^{(1)} + S_z^{(2)}\right)\chi_1\chi_2 = \left(S_z^{(1)}\chi_1\right)\chi_2 + \chi_1\left(S_z^{(2)}\chi_2\right) \\ &= (\hbar m_1\chi_1)\chi_2 + \chi_1(\hbar m_2\chi_2) = \hbar\underbrace{(m_1 + m_2)}_{m}\chi_1\chi_2 \end{aligned}$$

E.g. for particle 1, this can be up or down along the *z* axis.

$$\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$$

$$S_z\chi_1\chi_2 = \hbar\ m\ \chi_1\chi_2 \longrightarrow$$

$\uparrow\uparrow$: $m = 1;$

$\uparrow\downarrow$: $m = 0;$

$\downarrow\uparrow$: $m = 0;$

$\downarrow\downarrow$: $m = -1.$

What does this mean? There are 2 combinations with *m*=0? Use "lowering operators" (or "raising operators") to clarify and find special combinations.

Reminder:
$$\mathbf{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_- = S_-^{(1)} + S_-^{(2)}$$

$$S_-(\uparrow\uparrow) = (S_-^{(1)}\uparrow)\uparrow + \uparrow(S_-^{(2)}\uparrow)$$

$$= (\hbar\downarrow)\uparrow + \uparrow(\hbar\downarrow) = \hbar(\downarrow\uparrow + \uparrow\downarrow)$$

This suggests the sum **up down + down up** "emerges" as special.
Also note up-down is not the same as down-up ! The "order" of particles must be respected.

Not in book (but probably in HW):

$$S_-(\downarrow\uparrow + \uparrow\downarrow) = (S_-^{(1)} + S_-^{(2)})(\downarrow\uparrow + \uparrow\downarrow) =$$

$$= (\underbrace{S_-^{(1)}\downarrow}_{=\,0})\uparrow + (\underbrace{S_-^{(1)}\uparrow}_{(\hbar\,\downarrow)})\downarrow + \downarrow(\underbrace{S_-^{(2)}\uparrow}_{(\hbar\,\downarrow)}) + \uparrow(\underbrace{S_-^{(2)}\downarrow}_{=\,0}) =$$

$$= 2\,\hbar\downarrow\downarrow$$

This means that $\uparrow\uparrow$, $(\downarrow\uparrow + \uparrow\downarrow)$, and $\downarrow\downarrow$ are linked forming a set called the triplet (see (4.175) book, page 177)

This takes care of 3 states. How about number 4?

Make sure you understand this page and next, step by step.

In book not sufficient detail (probably in HW you will repeat):

$$S_-(\downarrow\uparrow - \uparrow\downarrow) = (S_-^{(1)} + S_-^{(2)})(\downarrow\uparrow - \uparrow\downarrow) =$$

$$= (\underbrace{S_-^{(1)}\downarrow}_{=0})\uparrow - (\underbrace{S_-^{(1)}\uparrow}_{(\hbar\downarrow)})\downarrow + \downarrow(\underbrace{S_-^{(2)}\uparrow}_{(\hbar\downarrow)}) - \uparrow(\underbrace{S_-^{(2)}\downarrow}_{=0}) =$$

$$= 0.$$

If you apply S_+ it gives zero as well. This means that $(\uparrow\downarrow - \downarrow\uparrow)$ is a state **alone** (we say is non degenerate) forming a singlet.

$$\left\{\begin{array}{ll} |1\,1\rangle & = \uparrow\uparrow \\ |1\,0\rangle & = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,-1\rangle & = \downarrow\downarrow \end{array}\right\} \quad s = 1$$

(triplet: 3 discrete projections)

$$\left\{|0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right\} \quad s = 0$$

(singlet: only 1 projection)

$|s\,m\rangle$

s denotes the total spin and *m* denotes the *z*-axis projection of the total spin.