

### 4.4.3 Addition of angular momenta (page 176)

Important: First time we go beyond one electron!

Suppose we have **TWO particles, each with spin  $\frac{1}{2}$ .**

Here we focus on the spin, not the actual relative motion.  
Could be 2 electrons or 1 electron and 1 proton ...

Because each spin can be up or down, overall, *regardless of real distance between electrons*, we have 4 cases:

$\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

We want to know: what is the **total spin**? Natural answers are  $1=1/2 + 1/2$  or  $0=1/2 - 1/2$ . **But why not other numbers?** Classically, any number between 1 and 0 is good.

This means operators of particle 1.  
Now we have to label each particle.

The total spin operator is:  $\mathbf{S} \equiv \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$

The z-component of the total spin operator is:

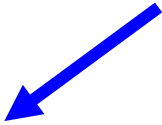
$$S_z = S_z^{(1)} + S_z^{(2)}$$

$\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

$$\begin{aligned} S_z \chi_1 \chi_2 &= (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2 = (S_z^{(1)} \chi_1) \chi_2 + \chi_1 (S_z^{(2)} \chi_2) \\ &= (\hbar m_1 \chi_1) \chi_2 + \chi_1 (\hbar m_2 \chi_2) = \hbar \underbrace{(m_1 + m_2)}_m \chi_1 \chi_2 \end{aligned}$$

E.g. for particle 1, this can be up or down along the z axis.

$$S_z \chi_1 \chi_2 = \hbar m \chi_1 \chi_2$$

$\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$   


$\uparrow\uparrow: m = 1;$   
 $\uparrow\downarrow: m = 0;$   
 $\downarrow\uparrow: m = 0;$   
 $\downarrow\downarrow: m = -1.$

What does this mean? There are 2 combinations with  $m=0$ ?  
 Use "lowering operators" (or "raising operators") to clarify  
 and find **special combinations**.

$$S_- = S_-^{(1)} + S_-^{(2)}$$

Reminder:

$$\mathbf{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned}
 S_-(\uparrow\uparrow) &= (S_-^{(1)} \uparrow) \uparrow + \uparrow (S_-^{(2)} \uparrow) \\
 &= (\hbar \downarrow) \uparrow + \uparrow (\hbar \downarrow) = \hbar(\downarrow\uparrow + \uparrow\downarrow)
 \end{aligned}$$

This suggests the sum **up down + down up** "emerges" as special.  
 Also note up-down is not the same as down-up! The "order" of  
 particles must be respected.

Not in book (but probably in HW):

$$\begin{aligned} S_- (\downarrow\uparrow + \uparrow\downarrow) &= (S_-^{(1)} + S_-^{(2)}) (\downarrow\uparrow + \uparrow\downarrow) = \\ &= \underbrace{(S_-^{(1)} \downarrow)}_{=0} \uparrow + \underbrace{(S_-^{(1)} \uparrow)}_{(\hbar \downarrow)} \downarrow + \downarrow \underbrace{(S_-^{(2)} \uparrow)}_{(\hbar \downarrow)} + \uparrow \underbrace{(S_-^{(2)} \downarrow)}_{=0} = \\ &= 2\hbar \downarrow\downarrow \end{aligned}$$

This means that  $\uparrow\uparrow$ ,  $(\downarrow\uparrow + \uparrow\downarrow)$ , and  $\downarrow\downarrow$  are linked forming a set called the **triplet** (see (4.175) book, page 177)

This takes care of 3 states. How about number 4?

Make sure you understand this page and next, step by step.

In book not sufficient detail (probably in HW you will repeat):

$$\begin{aligned} S_- (\downarrow\uparrow - \uparrow\downarrow) &= (S_-^{(1)} + S_-^{(2)}) (\downarrow\uparrow - \uparrow\downarrow) = \\ &= \underbrace{(S_-^{(1)} \downarrow)}_{=0} \uparrow - \underbrace{(S_-^{(1)} \uparrow)}_{(\hbar \downarrow)} \downarrow + \downarrow \underbrace{(S_-^{(2)} \uparrow)}_{(\hbar \downarrow)} - \uparrow \underbrace{(S_-^{(2)} \downarrow)}_{=0} = \\ &= 0. \end{aligned}$$

If you apply  $S_+$  it gives zero as well. This means that  $(\uparrow\downarrow - \downarrow\uparrow)$  is a state **alone** (we say is non degenerate) forming a **singlet**.

$$\left\{ \begin{array}{l} |1\ 1\rangle = \uparrow\uparrow \\ |1\ 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\ -1\rangle = \downarrow\downarrow \end{array} \right\} \quad s = 1 \quad \text{(triplet: 3 discrete projections)}$$

$$\left\{ |0\ 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\} \quad s = 0 \quad \text{(singlet: only 1 projection)}$$

$|s\ m\rangle$   $s$  denotes the total spin and  $m$  denotes the z-axis projection of the total spin.