What we called the "triplet" has three states, but is it truly a state of total spin 1?

Consider the **total** spin operator:

The complicated part

$$S^2 = (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) \cdot (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + \underbrace{2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}}$$

Example for state "up down":

$$\begin{aligned}\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}(\uparrow\downarrow) &= (S_x^{(1)} \uparrow)(S_x^{(2)} \downarrow) + (S_y^{(1)} \uparrow)(S_y^{(2)} \downarrow) + (S_z^{(1)} \uparrow)(S_z^{(2)} \downarrow) \\ &= \left(\frac{\hbar}{2}\downarrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(\frac{i\hbar}{2}\downarrow\right)\left(\frac{-i\hbar}{2}\uparrow\right) + \left(\frac{\hbar}{2}\uparrow\right)\left(\frac{-\hbar}{2}\downarrow\right) \\ &= \frac{\hbar^2}{4}(2\downarrow\uparrow - \uparrow\downarrow)\end{aligned}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0 \\ i\end{pmatrix} = \frac{\hbar}{2} i \begin{pmatrix}0 \\ 1\end{pmatrix}$$

Similarly: $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}(\downarrow\uparrow) = \frac{\hbar^2}{4}(2\uparrow\downarrow - \downarrow\uparrow)$

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}|1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2\downarrow\uparrow - \uparrow\downarrow + 2\uparrow\downarrow - \downarrow\uparrow) = \frac{\hbar^2}{4}|1\,0\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$S^2|1\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|1\,0\rangle = 2\hbar^2|1\,0\rangle$$

Total spin squared

$(S^{(1)})^2$ $\quad (S^{(2)})^2$ $\quad 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$

S(S+1) = 1 (1+1) =2
for total spin

s(s+1) = 1/2 (1/2 + 1) =3/4
for individual spins

Same story with "singlet" (please complete steps):

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} |0\,0\rangle = \frac{\hbar^2}{4} \frac{1}{\sqrt{2}} (2 \downarrow\uparrow - \uparrow\downarrow - 2 \uparrow\downarrow + \downarrow\uparrow) = -\frac{3\hbar^2}{4} |0\,0\rangle$$

Just a sign difference with previous page!

$$S^2 |0\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - 2\frac{3\hbar^2}{4} \right) |0\,0\rangle = 0$$

S(S+1) = 0 (0+1) =0 for total spin

Also left as exercise, application of total spin over "up up" $|1\,1\rangle$ and "down down" $|1\,-1\rangle$.

$|s\ m\rangle$

Then, we have confirmed that indeed we form a **triplet** and a **singlet**, out of two spins $\frac{1}{2}$.

$$\left\{\begin{array}{ll} |1\ 1\rangle & = \uparrow\uparrow \\ |1\ 0\rangle & = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\ {-1}\rangle & = \downarrow\downarrow \end{array}\right\} \quad s = 1 \qquad \text{triplet}$$

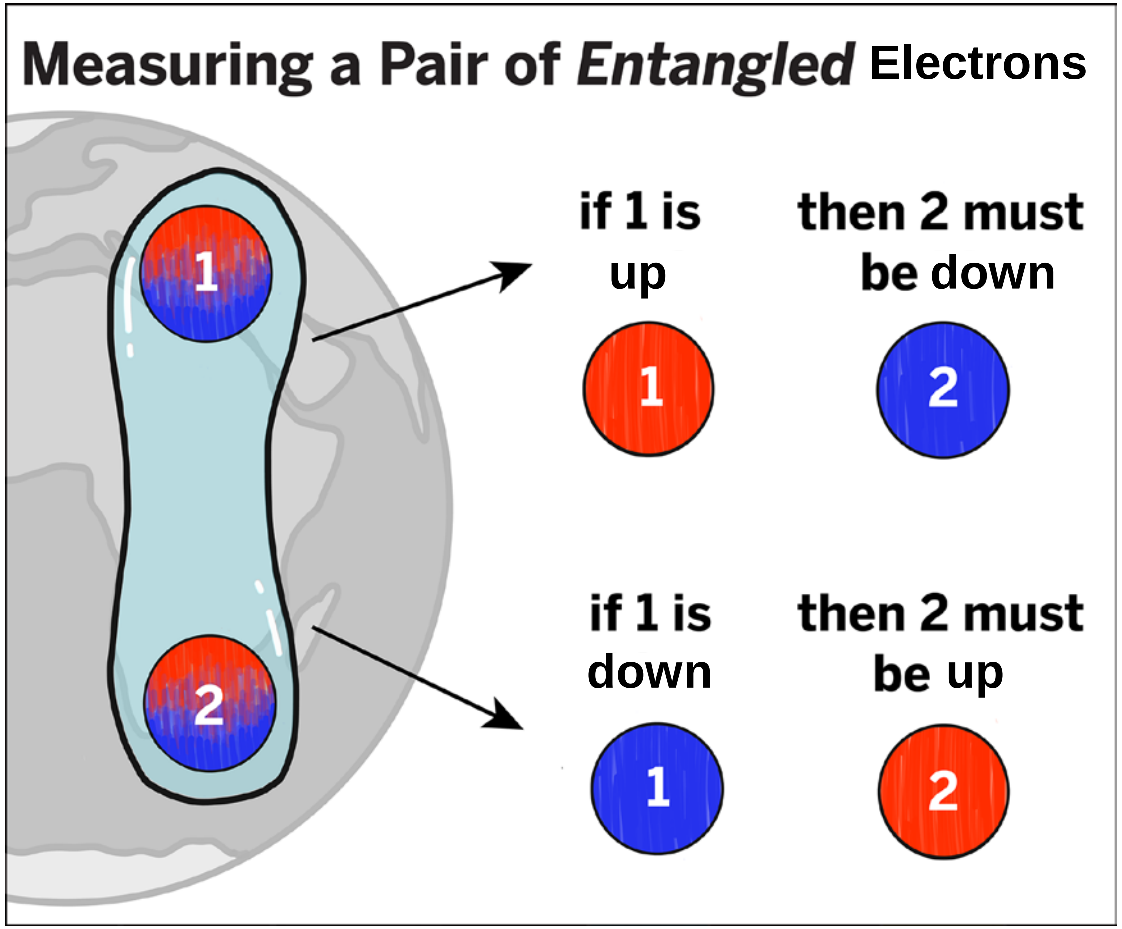
$$\left\{|0\ 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right\} \quad s = 0 \qquad \text{singlet}$$

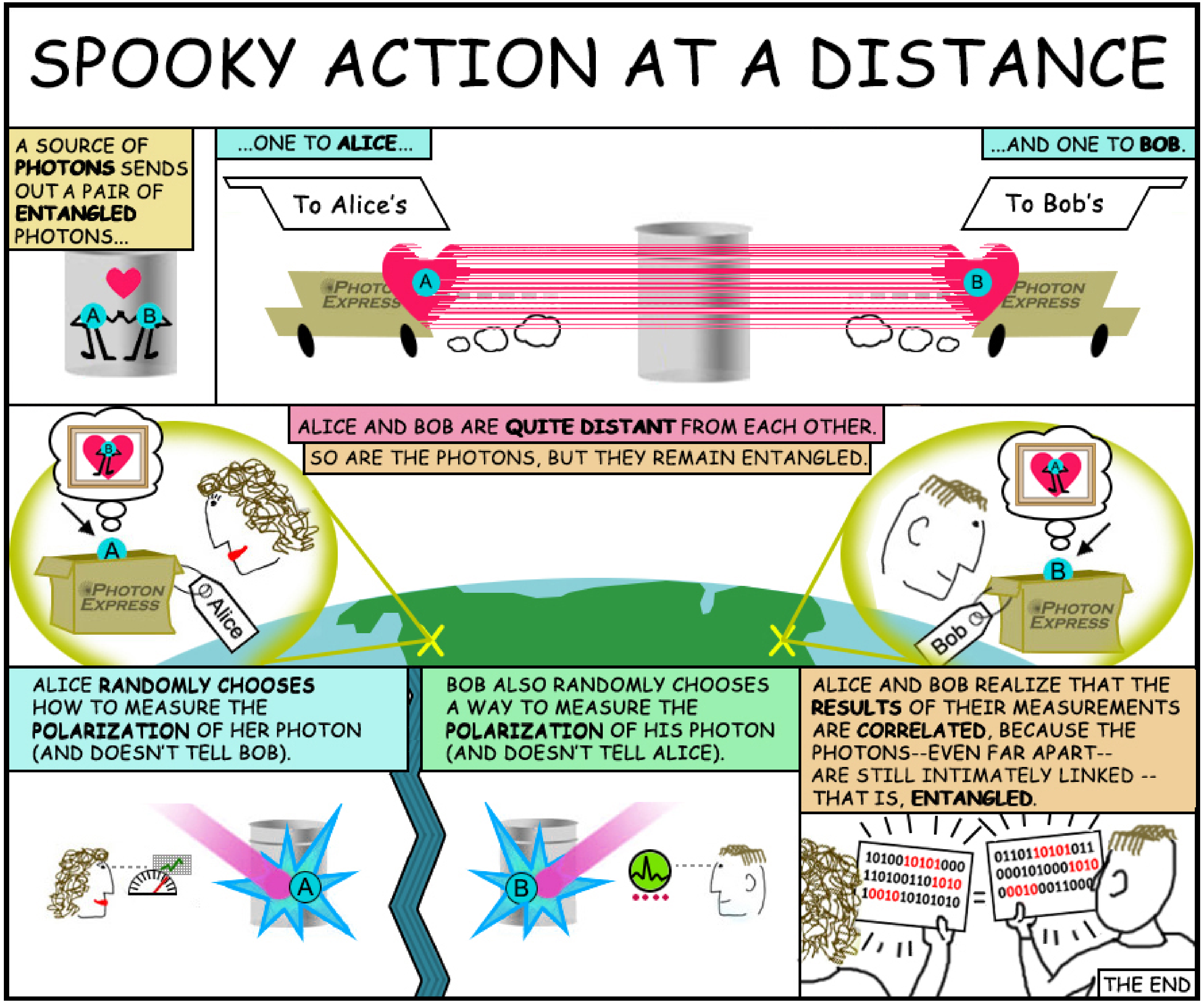
Side comment: Consider the **singlet**. The two spins $\frac{1}{2}$ are "quantum entangled" because while always they point in opposite directions, it could be up down **or** it could be down up. This is another example of "quantum superposition", which occurs even for 1 spin, as in spinor $\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\left\{ |0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\} \quad s = 0$$

Suppose the two electrons are separated by light years and, *big if*, suppose you somehow keep the entanglement i.e. avoid interacting with the surrounding i.e. avoid "decoherence" when you separate them over such long distance.

If you measure and find "spin up", then for sure you know that the other particle, lights years away, has spin down. Einstein was not happy, called it "spooky action at a distance". Yet, it appears to be true (Nobel 2022).


Measuring a Pair of *Entangled* Electrons
1
2
if 1 is up
then 2 must be down
1
2
if 1 is down
then 2 must be up
1
2


SPOOKY ACTION AT A DISTANCE
A SOURCE OF PHOTONS SENDS OUT A PAIR OF ENTANGLED PHOTONS...
A
B
...ONE TO ALICE...
...AND ONE TO BOB.
To Alice's
To Bob's
PHOTON EXPRESS
A
B
PHOTON EXPRESS
ALICE AND BOB ARE QUITE DISTANT FROM EACH OTHER.
SO ARE THE PHOTONS, BUT THEY REMAIN ENTANGLED.
A
PHOTON EXPRESS
Alice
B
PHOTON EXPRESS
Bob
ALICE RANDOMLY CHOOSES HOW TO MEASURE THE POLARIZATION OF HER PHOTON (AND DOESN'T TELL BOB).
BOB ALSO RANDOMLY CHOOSES A WAY TO MEASURE THE POLARIZATION OF HIS PHOTON (AND DOESN'T TELL ALICE).
ALICE AND BOB REALIZE THAT THE RESULTS OF THEIR MEASUREMENTS ARE CORRELATED, BECAUSE THE PHOTONS--EVEN FAR APART--ARE STILL INTIMATELY LINKED -- THAT IS, ENTANGLED.
A
B
10100101010000
110100110101010
100010101101010
=
01101101010101011
0001010001010
00010001100
THE END

Not in book (counting of states for 2 and 3 spins):

↑↑ ↑↓ ↓↑ ↓↓

2^2=4 states sort of random

↑↑

$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

↓↓

$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

4 states grouped as S=1 (3) and S=0 (1)

Note: after finding S=1, there was only 1 state left, thus **had to be singlet** and **had to be orthogonal**, thus fixing the "-"

↑↑↑

↑↑↓ , ↑↓↑ , ↓↑↑

↑↓↓ , ↓↑↓ , ↓↓↑

↓↓↓

2^3 =8 states sort of random

↑↑↑ , S_↑↑↑, S^2_↑↑↑, S^3_↑↑↑

4 states form S total 3/2

The 4 states left form TWO S total ½ states (not triplet singlet).

3/2 ⊕ 1/2 ⊕ 1/2 (more about this in a few pages)

Not in book and FYI only:

Spins can interact among themselves, not only with external magnetic fields.

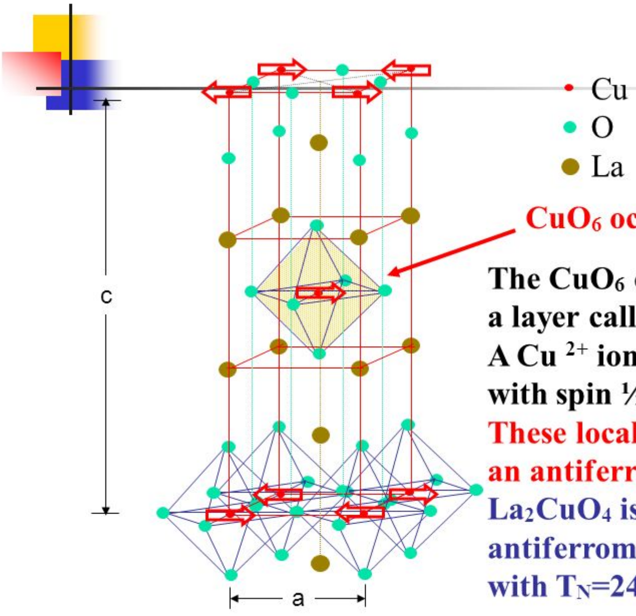
It is as if other spins with label "*j*" produce an **effective** magnetic field on the spin "*i*" you are looking at. This is the famous Heisenberg model:

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

Ground state? Number of states grows like 2^N (=2,4,8,16, …)

Record done exactly N~ 40.
2^{40} = 1,099,511,627,776 states
("funny note": US debt is even larger).

Crystal structure of undoped La_2CuO_4



The CuO_6 octahedrons form a layer called a CuO_2 layer. A Cu^{2+} ion has one d hole with spin ½.
These localized spins form an antiferromagnetic phase.
La_2CuO_4 is the antiferromagnetic insulator with T_N=240K.

WITHOUT PROOF, this is what happens when you combine a spin s_1 and a spin s_2 (each individually 0, 1/2, 1, 3/2, …).

The total spin s of the combination can be Eq.(4.182):

$$s = (s_1 + s_2),\ (s_1 + s_2 - 1),\ (s_1 + s_2 - 2),\ \ldots,\ |s_1 - s_2|$$

Example 1: for $s_1 = 1/2$ and $s_2 = 1/2$, then s runs from $s_1 + s_2 = 1$ to $|s_1 - s_2| = 0$, with nothing in between. We already confirmed this is true.

Example 2: for $s_1 = 3/2$ and $s_2 = 2$, then s runs from $s_1 + s_2 = 7/2$ to $|s_1 - s_2| = 1/2$, with 5/2 and 3/2 in between.

Example 3: this unproven theorem holds also for the addition of orbital angular momentum l and spin s. For l=2 and s=1/2, then total j runs from for $l+s$ = 5/2 to $|l-s|$= 3/2, with nothing in between.

Example 4: if you have three particles with s_1=1/2, s_2=1/2, and s_3=1/2, then first you add two, such as s_1 and s_2, finding $s_{partial}$ = 1,0 and then add $s_{partial}$ with s_3, finding 3/2,1/2 (for $s_{partial}$ =1) and another 1/2 (for $s_{partial}$=0). So there are two independent combinations with total spin $\frac{1}{2}$. We already saw that in previous pages: $3/2 \oplus 1/2 \oplus 1/2$

With the foundation given already, it should be easy for you to learn from the book the **Clebsch-Gordan coefficients** that are often needed in nuclear physics.