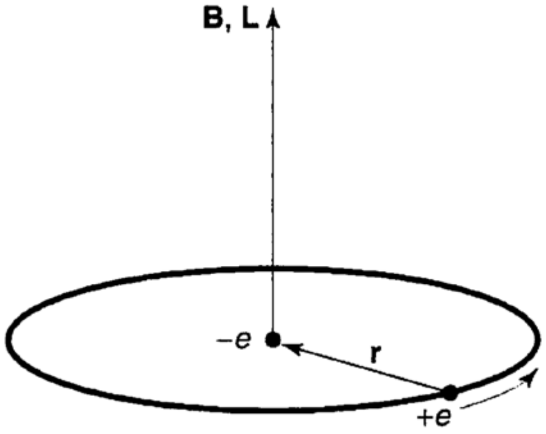
7.3.2 Spin-orbit coupling:



From the perspective of the electron, the proton is orbiting around:

From classical electromagnetism a loop of current creates a magnetic field at the center:

$$B = \frac{\mu_0 I}{2r}$$

$$c = 1/\sqrt{\epsilon_0 \mu_0}$$

where the permeability μ_0 of free space and the permittivity of free space ε_0 are related via:

Side comment: I will send video about this amazing formula.

The current relates with the period T via:

$$I = e/T$$

$\omega = 2\pi/T$

and the period T relates with the angular momentum L via:

$$L = I\,\omega = m\,r^2\,2\pi/T$$

From study of spins in magnetic fields, Ch. 4, in general:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

From previous study of spin:

$$\mu_e = -\frac{e}{m}\mathbf{S}$$

From previous page:

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L}$$

Then, overall, classically we obtain the correction called spin-orbit coupling:

$$H = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{S}\cdot\mathbf{L}$$

Repeating via rigorous math (including the fact that the electron inertia system is accelerating, etc.) leads just to a factor 2 difference:

$$H'_{\mathrm{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{S}\cdot\mathbf{L}$$

Important statement, without proof: Because of the **S.L** term, the Hamiltonian no longer commutes with S_z (quantum number m_s) and L_z (quantum number m_l) but still commutes with L^2 and S^2 and the total angular momentum J^2 and its projection J_z (quantum number m_j).

$$\mathbf{J} \equiv \mathbf{L} + \mathbf{S}$$

$$J^2 = (\mathbf{L} + \mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S}$$

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2)$$

Change of basis required: from basis with (n, l, s, m_l, m_s) as quantum numbers to basis with (n, l, s, j, m_j) as quantum numbers.

$$\frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$$

Fixed to 3/4

Moreover, the expectation value of $1/r^3$ using H wave functions is:

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{l(l+1/2)(l+1)n^3a^3}$$

$$\langle H'_{so} \rangle = \langle \left(\frac{e^2}{8\pi\epsilon_0} \right) \frac{1}{m^2c^2r^3} \mathbf{S}\cdot\mathbf{L} \rangle$$

$$\frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{l(l+1/2)(l+1)n^3a^3}$$

First order in perturbation theory

$$E^1_{so} = \langle H'_{so} \rangle = \frac{e^2}{8\pi\epsilon_0} \frac{1}{m^2c^2} \frac{(\hbar^2/2)[j(j+1) - l(l+1) - 3/4]}{l(l+1/2)(l+1)n^3a^3}$$

Eqs. from Ch 4 for the Bohr radius *a* and for E_n :

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

$$E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2}$$

$$E^1_{so} = \frac{(E_n)^2}{mc^2} \left\{ \frac{n[j(j+1) - l(l+1) - 3/4]}{l(l+1/2)(l+1)} \right\}$$

In spite of their very different origins, the relativistic and spin-orbit corrections have the same number in front. I.e. they are of the "same order".

Adding both terms naively leads to a long expression

$$E_r^1 = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{l+1/2} - 3\right] + E_{so}^1 = \frac{(E_n)^2}{mc^2}\left\{\frac{n[j(j+1) - l(l+1) - 3/4]}{l(l+1/2)(l+1)}\right\}$$

Consider j=l+1/2 first. Then j(j+1) = (l+1/2)(l+3/2). Careful algebra (HW problem) leads to

$$E_{fs}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

Consider now j=l-1/2. Then j(j+1) = (l-1/2)(l+1/2). Careful algebra (HW problem) leads to the **same**

$$E_{fs}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

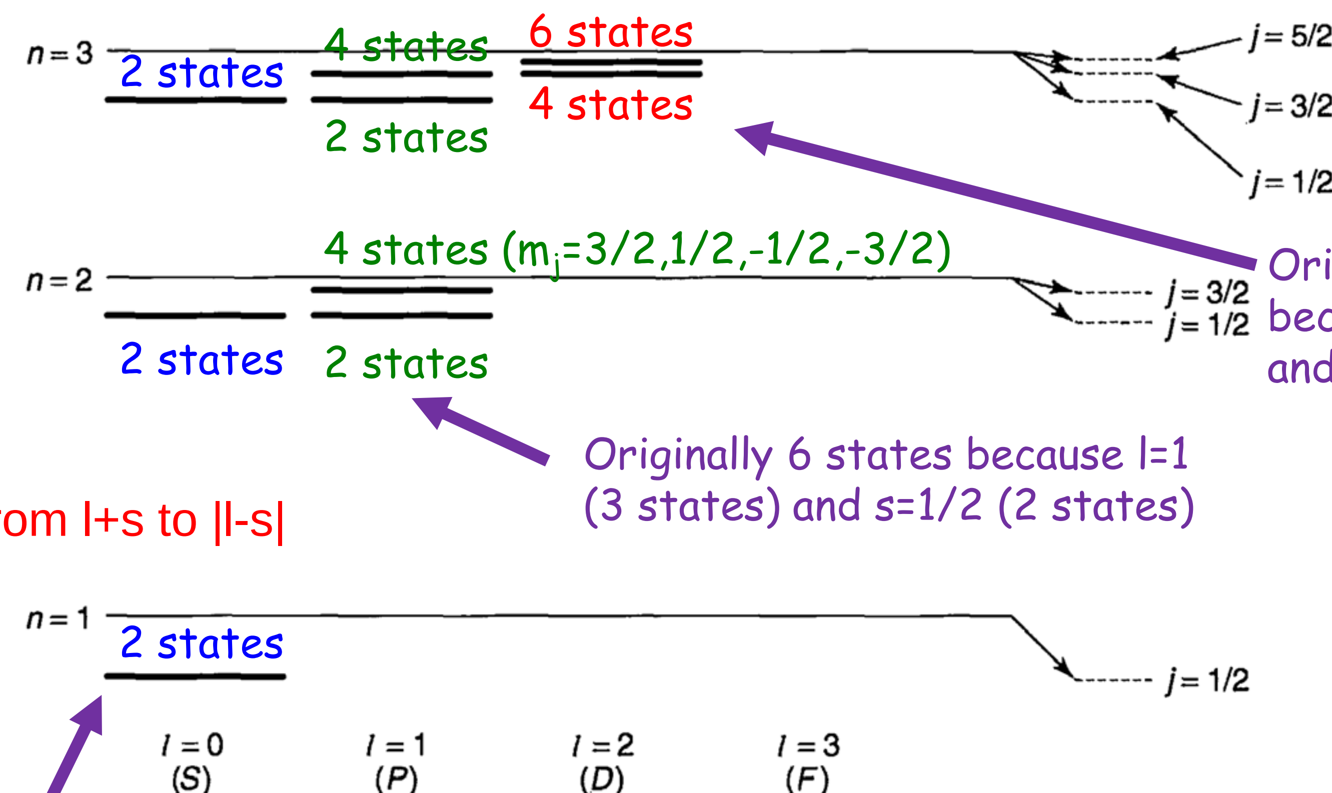
In HW problem you will show that order 0 + order 1 gives:

$$E_{nj} = -\frac{13.6\,\text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]$$

where

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \cong \frac{1}{137.036}$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$



Originally 10 states because l=2 (5 states) and s=1/2 (2 states)

Originally 6 states because l=1 (3 states) and s=1/2 (2 states)

j runs from l+s to |l-s|

Originally 2 states because l=0 is 1 state but spin s=1/2 makes it 2 states.

NOTE: this is for the H atom. For He and beyond the e-e repulsion plays a more important role than the fine structure.

Counting of states repeated for better clarity:

If we consider the "p levels", they have angular momentum $l=1$. The spin is $s=1/2$. Then, the total angular momentum j can be $1+1/2=3/2$ and $1-1/2=1/2$.

The original $l=1$ is 3 states $m_l=1,0,-1$. The original spin $s=1/2$ is 2 states $m_s=1/2,-1/2$. Total = 6 states.

The $j=3/2$ is 4 states $m_j = 3/2, 1/2, -1/2, -3/2$. The $j=1/2$ is 2 states $m_j = 1/2, -1/2$. Total = 6 states.

The original 6 degenerate states are simply mixed in special linear combinations to become states with j sharply defined and different j projections.

We never create or destroy states by adding perturbations. They just rearrange.