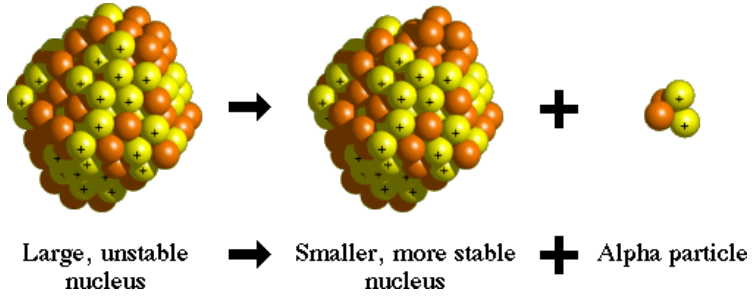
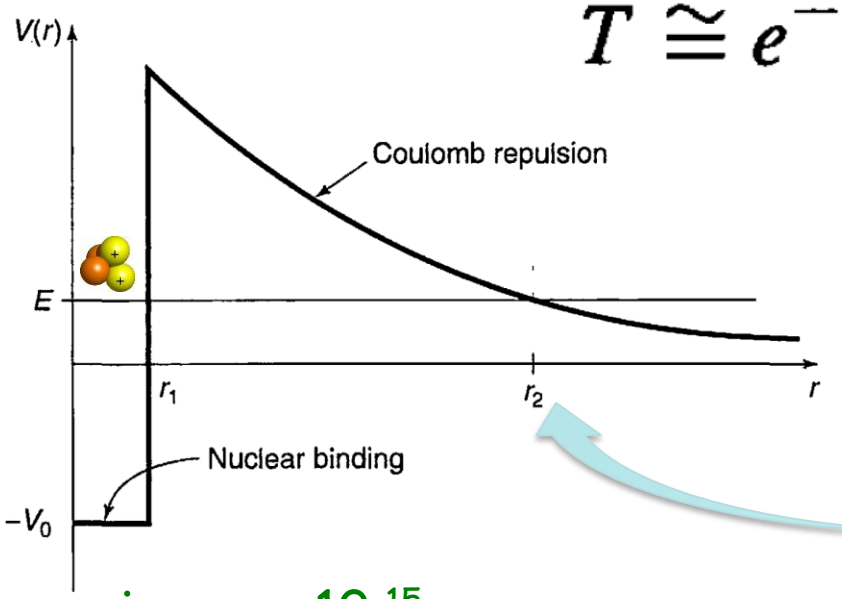
Famous example: Gamow's theory of alpha decay (1928). First use of QM in nuclear.





$$T \cong e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$

$$\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r_2} = E$$

Z=leftover nucleus

size $r_1 \sim 10^{-15}$ m

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2}\sqrt{2m\left(\underbrace{\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r}}_{V(r)} - E\right)}\,dr$$

Turns out, the integral can be done exactly (see book), and moreover it can be simplified considerably if $r_1 \ll r_2$

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2}\sqrt{2m\left(\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r} - E\right)}\,dr \quad \rightarrow \quad K_1\frac{Z}{\sqrt{E}} - K_2\sqrt{Zr_1}$$

Full integral

After some approximations

where

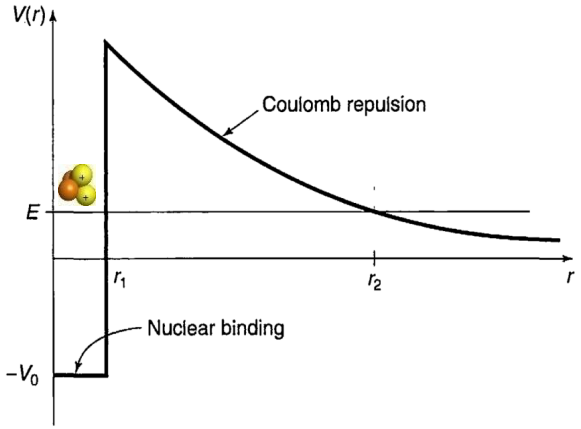
$$K_1 \equiv \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\pi\sqrt{2m}}{\hbar} = 1.980\ \text{MeV}^{1/2}$$

$$K_2 \equiv \left(\frac{e^2}{4\pi\epsilon_0}\right)^{1/2}\frac{4\sqrt{m}}{\hbar} = 1.485\ \text{fm}^{-1/2}$$

1 fm = 10^{-15} m is the size of a typical nucleus

As "m" we use the mass of an alpha particle ~ 4 proton masses

Z is the positive charge of the nucleus

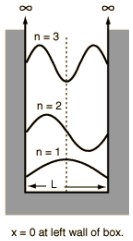


If alpha particles have an average velocity "v" inside the well, then to travel from r=0 to $r=r_1$ it takes $t=r_1/v$, i.e. hits the walls with a period $2r_1/v$. At each collision the probability of **remaining trapped** is high $e^{+2\gamma}$ (or prob. of escape is low $e^{-2\gamma}$)

The **lifetime** then is $\tau = (2r_1/v)\, e^{+2\gamma}$.

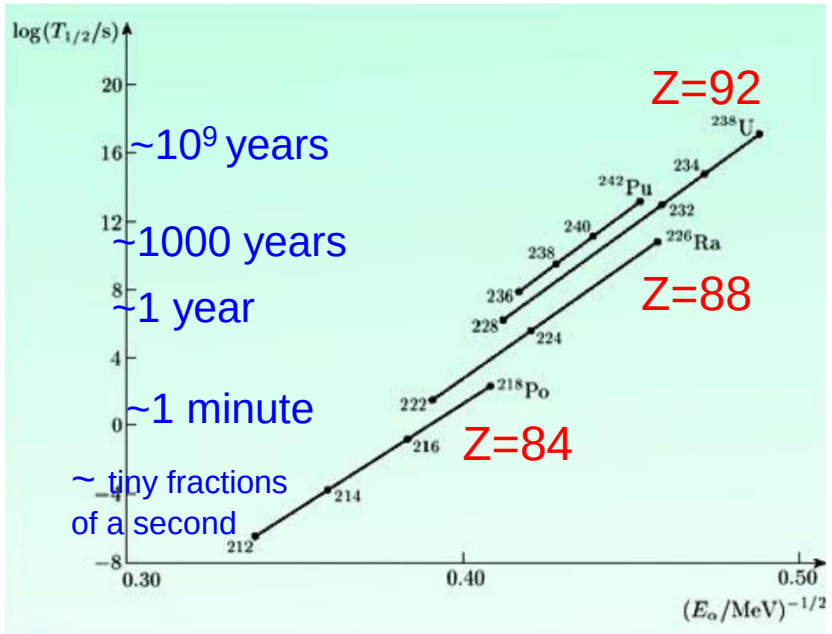
Then, $\ln(\tau) = \ln(2r_1/v) + 2\gamma$ with

$$\gamma \sim K_1 \frac{Z}{\sqrt{E}} - K_2 \sqrt{Z r_1}$$



Note: the energy of the alpha particle is not arbitrary but resembles that of a square well.

Experiments confirm that lifetime depends ~ linearly on 1/sqrt{E_α} on a range of lifetimes from 10^9 years to tiny fractions of seconds!
(Geiger-Nuttall law)



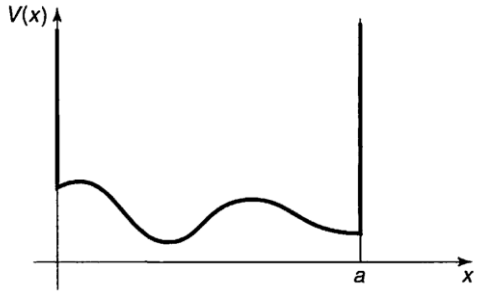
You may remember the video I asked you to watch last semester about how the sun proceeds with nuclear fusion … even though it is not hot enough to overcome the Coulombic barrier for the fusion of four protons into a He nucleus (plus two neutrons plus two positrons ejected).

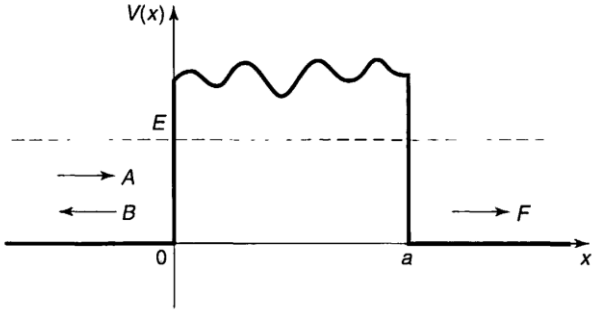
https://www.youtube.com/watch?v=lQapfUcf4Do

The Ch 9 portion of the lecture ended here. The next three pages (page 6,7,8) are only for completeness. It is not material that you need to know for Test 3 (final exam).Read if you like only.

9.3: The connection region (not for Test3)

In many examples we use the WKB approximation in cases V(x) has vertical walls.

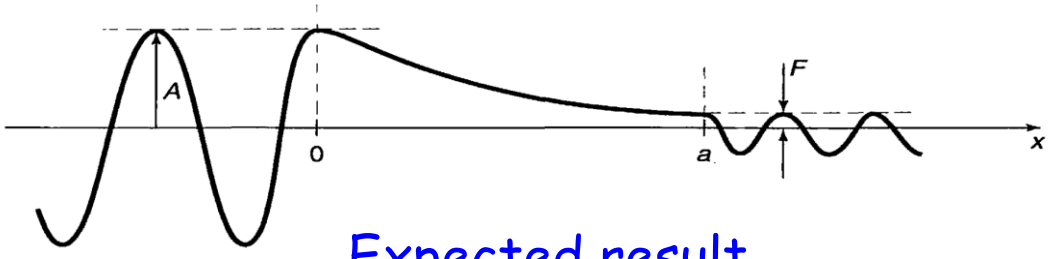




But in most real situations, this is not the case, such as in alpha decay. We may try the "usual" procedure:

$$\psi(x) \cong \begin{cases} \dfrac{1}{\sqrt{p(x)}}\left[Be^{\frac{i}{\hbar}\int_x^0 p(x')\,dx'} + Ce^{-\frac{i}{\hbar}\int_x^0 p(x')\,dx'}\right], & \text{if } x < 0, \\ \dfrac{1}{\sqrt{|p(x)|}} De^{-\frac{1}{\hbar}\int_0^x |p(x')|\,dx'}, & \text{if } x > 0. \end{cases}$$

Naively we may simply be tempted to try to match coefficients at the boundary.



Expected result

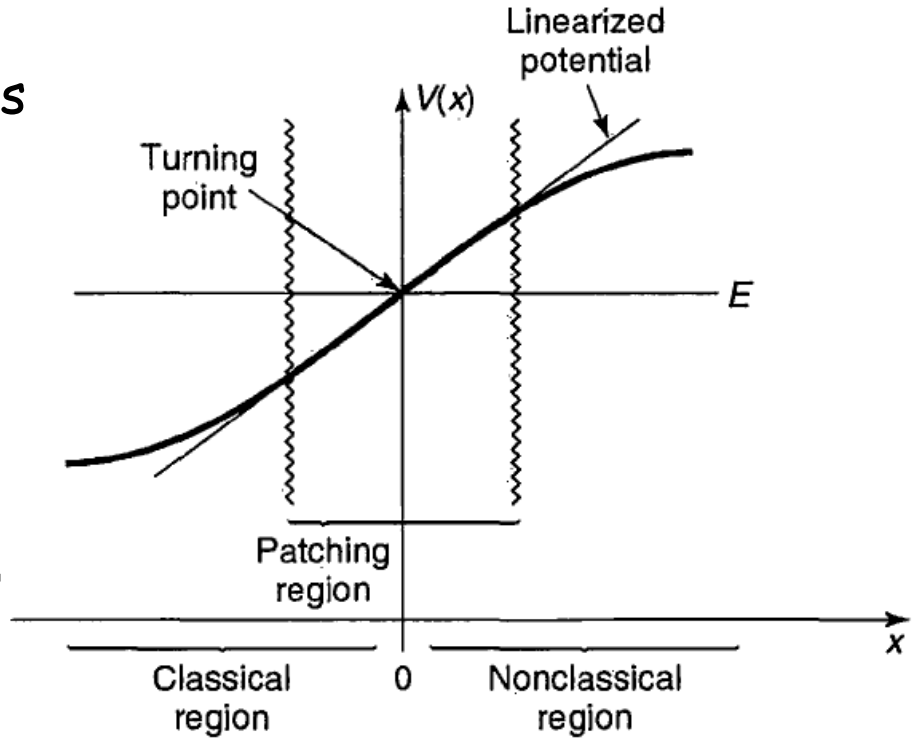
However, at exactly the "x" where we switch from classical to non-classical then p(x) = V(x)-E is zero. Then, WKB wave functions explode. Not realistic!

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar}\int |p(x)|\,dx}$$

In practice a "patching procedure" is followed, where a "third region" is introduced where the potential is linearized

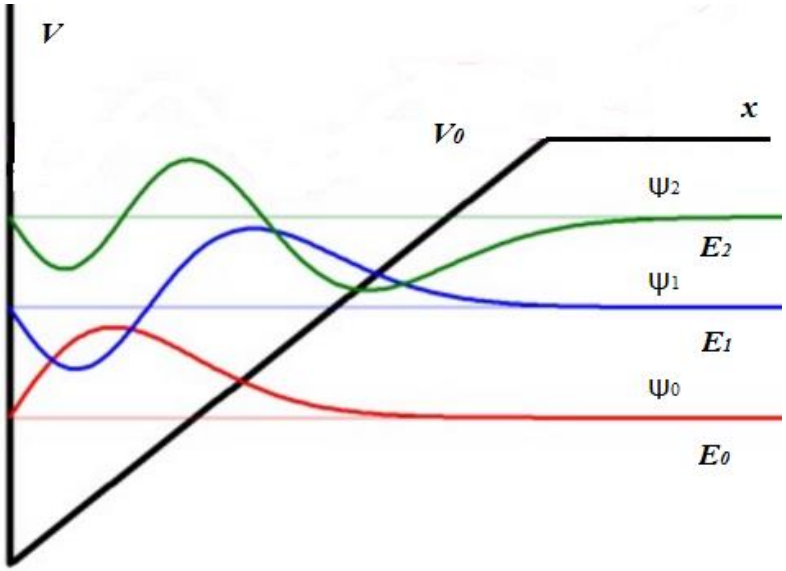
$$V(x) \cong E + V'(0)x$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_p}{dx^2} + [E + V'(0)x]\psi_p = E\psi_p$$



A problem with a linear potential is exactly solvable and leads to the Airy functions, complicated functions usually given in an integral form. They are oscillatory on one side and exponential on the other.

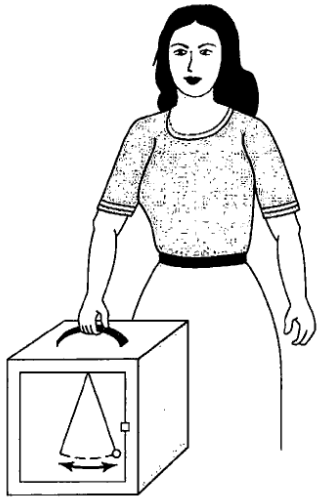
If we had a sharp wall on one side (not the actual problem at hand) the shape of the Airy functions is as shown (leading to bound states):



The WKB patching procedure would be too complicated to describe in detail, just be aware of its existence.

Section 11.5: The Adiabatic Approximation

Classical analog: a pendulum with zero friction oscillating in a plane. If abruptly you move the box, the pendulum will have a complicated motion.



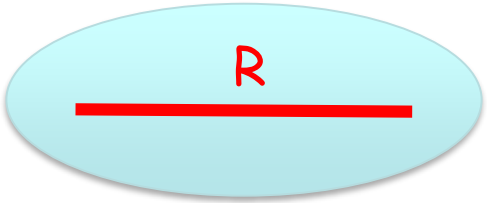
But if you "very slowly" rotate the box, the plane of oscillation will slowly rotate as well.

A very slow change in the boundary conditions of a problem defines an "adiabatic" process.

Often there are two characteristic times competing. One is internal, T_i, like the period of the pendulum that depends on gravity "g" and length of string "L". The other one is external, T_e, related in this example to the speed of rotation of the box. Adiabatic means $T_e \gg T_i$.

In QM, this is often used in describing oscillations of molecules. Like we did for the H_2^+ ion: we fixed R, the distance between hydrogens, and solved the electron problem. The results depend on R of course. Then, by making R=R(t) we could have found the frequencies of oscillations of the atoms, precursor of finding "phonons" in crystals.

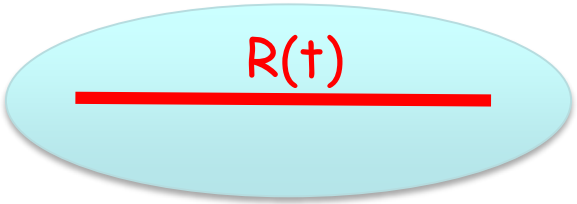
$\psi = A\,[\psi_0(r_1) + \psi_0(r_2)]$ at R fixed



Same type of wave function

$\psi = A\,[\psi_0(r_1) + \psi_0(r_2)]$

simply at new R (t)

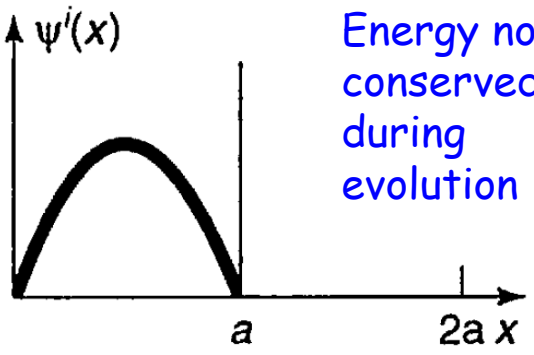


Solving the electronic problem first, and then addressing the oscillations in a fixed electronic cloud is called the **Born-Oppenheimer approximation**.

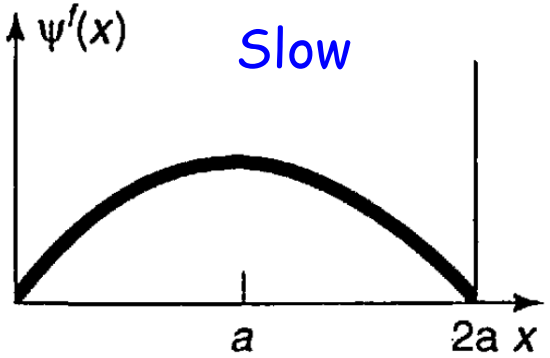
More specifically, if the electron was initially in the ground state, **it remains in the ground state** that smoothly changes its shape as R changes slowly. **This uses the Adiabatic Approximation !**

In general: if the particle is initially at eigenstate n-th of initial *H*, then in an adiabatic problem it remains in the eigenstate nth of the final *H*.

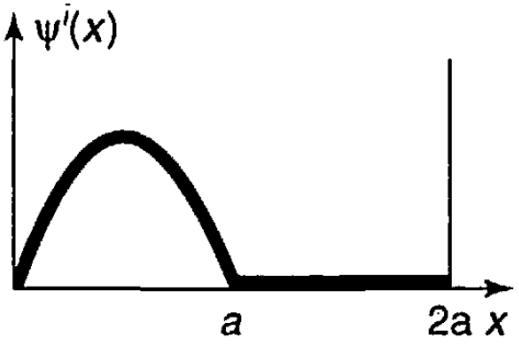
Now consider an **abrupt** switch from a to 2a.



Energy not conserved during evolution



Slow



$$\psi^i(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$$

$$\psi^f(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{\pi}{2a}x\right)$$

The initial wave function is neither even nor odd, thus it will have a nonzero overlap with all the eigenstates!

Moreover, after the wall abruptly is moved to 2a, the energy is **conserved** in the new problem. The electron will never be 100% in the ground state.

Definitely the wave function $\psi(t)$ will NOT be

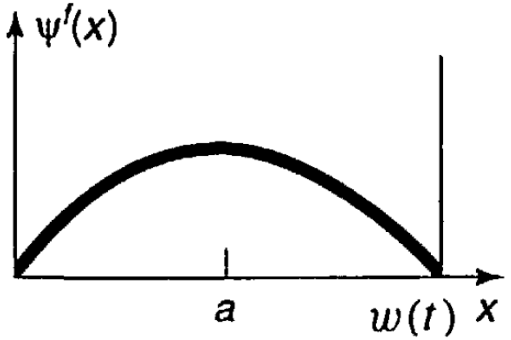
$$\psi^f(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{\pi}{2a}x\right)$$

The adiabatic theorem is complicated to prove but we will try. Specifically it says that in an adiabatic process the eigenstates – assume a discrete energy spectrum so that you can follow each one - evolve as:

$$\Psi_n(t) = e^{i\theta_n(t)} e^{i\gamma_n(t)} \psi_n(t)$$

For example, consider the square well again, with a wall moving with constant velocity *v*.

$$w(t) \equiv a + vt$$



$$\psi_n(x,t) = \sqrt{\frac{2}{w}} \sin\left(\frac{n\pi}{w} x\right)$$

In addition, there are two phase factors, according to the theorem: the dynamical and the geometrical (related to the Berry phase).