Normalization

Based on the statistical interpretation of $|\psi(x,t)|^2$, its integral has to be 1 because the particle must be somewhere.

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = 1$$

Thus, normalizing to 1 is just common sense.

If we are given a not normalized wave function f(x,t), we simply choose a multiplicative constant *A* such that

$$|A|^2 \int_{-\infty}^{+\infty} |f(x,t)|^2 \, dx = 1$$

The normalization is up to a constant phase factor that has no physical importance.

Notes: If $\psi=0$, then the integral can never be 1.
If the integral $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx$ diverges it cannot be normalized.

For all this to make sense, once we normalize to 1 at t=0 the normalization must remain. Otherwise particles will be created or vanished varying time. Is this true?

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}|\Psi(x,t)|^2\,dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

But $$\underbrace{\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi}_{} \rightarrow \frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

Multiplying all terms in Sch. Eq. by $-i/\hbar$

The terms with V cancel out.

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]$$

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \int_{-\infty}^{+\infty}\cancel{\frac{\partial}{\partial x}}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]\cancel{dx}$$

$$= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{+\infty} = 0$$

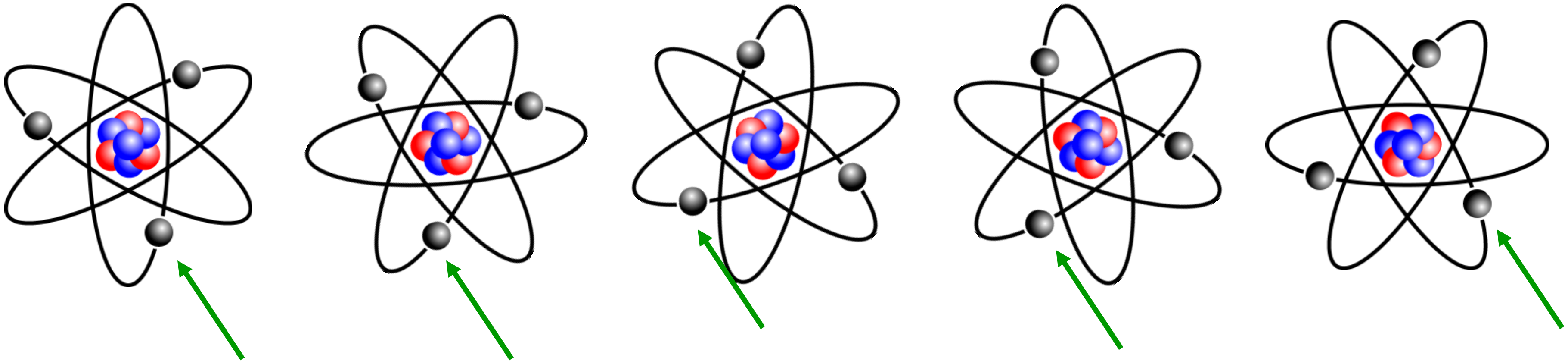
If $\psi \to 0$ as $x \to (\pm)$ infinity

If ψ is normalized at t=0, it remains normalized at all times. Crucial for all this to make sense!

Expectation value of x

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\Psi(x,t)|^2 \, dx$$

Interpretation: <x> is the average of measurements performed on an ensemble of identical systems.



Expectation value of momentum p

$$<v>= \frac{d\langle x\rangle}{dt} = \int x\frac{\partial}{\partial t}|\Psi|^2\,dx = \frac{i\hbar}{2m}\int \underbrace{x}_{f}\frac{\partial}{\partial x}\underbrace{\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)}_{g}dx$$

(See Chs 2 and 3)

By parts:

$$\int_a^b f\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}g\,dx + fg\Big|_a^b$$

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{2m}\int\left(\Psi^*\frac{\partial\Psi}{\partial x} - \underbrace{\frac{\partial\Psi^*}{\partial x}\Psi}\right)dx$$

By parts again:

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int \Psi^*\frac{\partial\Psi}{\partial x}\,dx$$

$$\boxed{\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx}$$

In summary, for *x* and *p* we find

$$\langle x \rangle = \int \Psi^* (x) \Psi \, dx \qquad \langle p \rangle = \int \Psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi \, dx$$

x "operator" is
just "multiply by x"

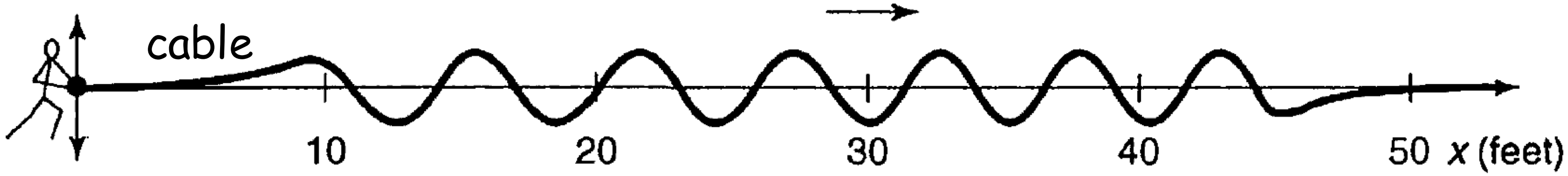
p is an "operator" and
is more complicated!

Many other operators are functions of x and p.
For instance, for the kinetic energy $T=p^2/2m$ use:

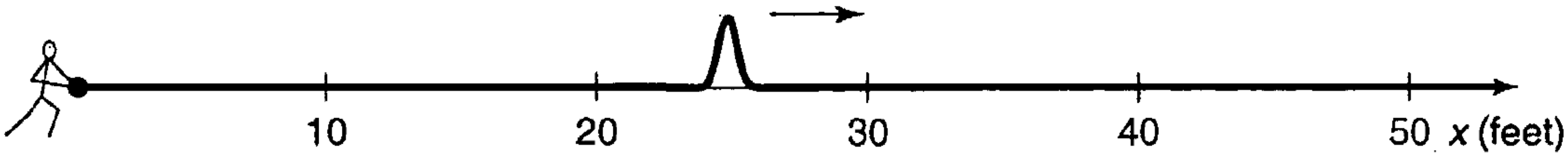
$$p^2 = \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

Preliminaries to the uncertainty principle

Caution: this is not an independent principle but it arises entirely from Sch. Eq. (see Ch. 3). Thus, if you do the calculations right, it is always satisfied. But intuitively it is interesting to discuss it.



Wavelength? Clear; Position? Unclear



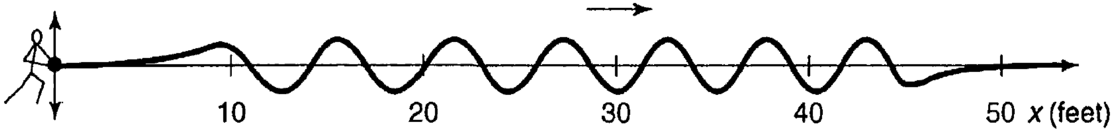
Wavelength? Unclear; Position? Clear

This is true for any wave-like phenomenon, thus it has to apply to the Sch. Eq. somehow as well.

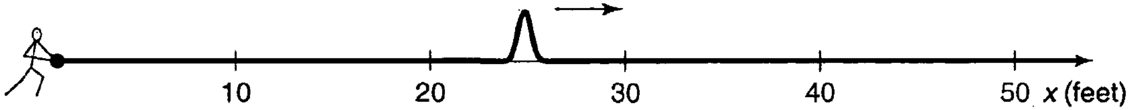
De Broglie formula (2 years before Sch. Eq.) said that electrons have wave-like features, like photons do:

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

Thus, if wavelength is known accurately, p is known accurately. If wavelength is unclear, p is unclear.



Momentum? Sharp
Position? Not sharp



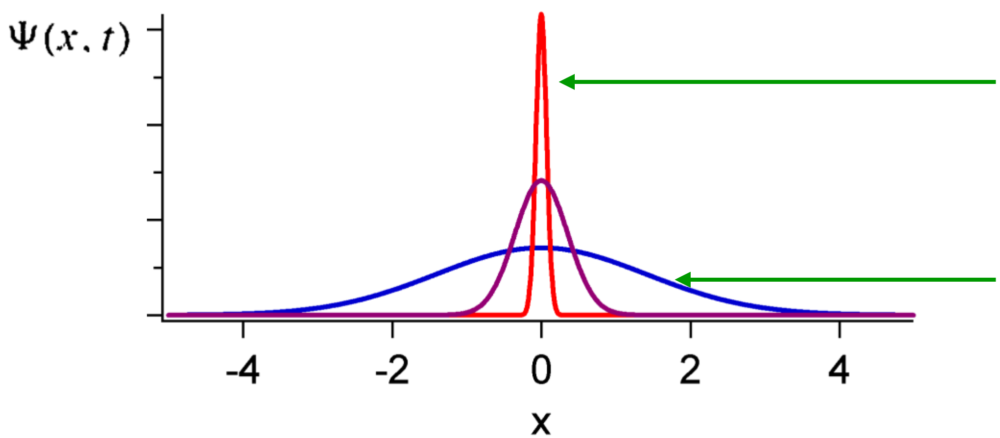
Momentum? Not sharp
(if you Fourier decomposed a spike, it has all k values!)
Position? Sharp

We will prove later (not a new law, but it's consequence of Sch. Eq.) that the standard deviations satisfy:

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$



Momentum? Not sharp
Position? Sharp

Momentum? Sharp
Position? Not sharp